An Initial Wave Packet

Our initial condition will be the so-called "minimum wave packet," having $\Delta x \Delta p = \frac{1}{2}\hbar$. Recall that, in general, the Heisenberg uncertainty relation implies that $\Delta x \Delta p \geq \frac{1}{2}\hbar$. It can be shown that, for a free particle, equality is attained if and only if the wavefunction has the form

$$\psi_{\rm min}(x) \;=\; \left[2\pi\Delta x^2\right]^{-\frac{1}{4}} \exp\left\{-\frac{(x-\bar{x})^2}{4\Delta x^2} + \frac{i\bar{p}x}{\hbar}\right\}.$$

It is easily shown that the expectation value of the position is

$$\langle x \rangle \;=\; \int_{-\infty}^{\infty} \psi^* x \psi \, dx \;=\; \bar{x},$$

and the expection of the momentum ($\hat{p} = -i\hbar\, \partial/\partial x$) is

$$\langle p \rangle \;=\; -\, i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x}\, dx \;=\; \bar{p}.$$

Similarly, the expectation energy of the wave packet energy is

$$\langle E \rangle \;=\; \frac{\bar{p}^2}{2m}.$$

Thus we can regard the wave packet as representing a particle centered on $\bar{x}$ and traveling to the right with momentum $\bar{p}$.

We will find that, as the packet propagates, it also spreads out in time, so

$$\Delta x^2(t) \;=\; \Delta x^2 + \frac{\hbar^2 t^2}{4m\Delta x^2}\,.$$

Nevertheless, this is our analog to the incoming particle stream in the time-independent scattering problem. We will choose

$$\psi(x,0) \;=\; \psi_{\rm min}(x)$$

for some specific choice of $\bar{x}$ and $\bar{p}$.