Von Neumann Stability Analysis for the FTCS diffusion scheme

In Exercise 6.1 we saw that explicit FTCS differencing of the advection equation is unstable for any time step. But the FTCS implementation seems to work for the diffusion equation, at least for some time steps. Why should that be? The von Neumann analysis tells us why.

The basic differencing scheme is

$$u_j^{n+1} = u_j^n + \alpha(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\,, \tag{1}$$

where $\alpha = D\Delta t/\Delta x^2$. As discussed previously, we look at the effect of this linear operator on a pure Fourier mode with wavenumber $k = 2\pi/\lambda$, with $e^{ikx_j} = e^{ikj\Delta x}$, and assume that the solution for the n-th time state is

$$u_j^n = \xi(k)^n\, e^{ikj\Delta x}.$$

Substituting this expression for u_j^n into Equation 1 allows us to determine which modes are stable. We find

$$\xi^{n+1}\, e^{ikj\Delta x} = \xi^n\, e^{ikj\Delta x} + \alpha\left(\xi^n\, e^{ik(j+1)\Delta x} - 2\xi^n\, e^{ikj\Delta x} + \xi^n\, e^{ik(j-1)\Delta x}\right).$$

Dividing through by $\xi^n\, e^{ikj\Delta x}$, this simplifies to

$$\begin{aligned}
\xi(k) &= 1 + \alpha\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) \\
&= 1 + 2\alpha(\cos k\Delta x - 1) \\
&= 1 - 4\alpha \sin^2 \tfrac{1}{2}k\Delta x.
\end{aligned} \tag{2}$$

Thus, if α is small, the second term is less than 1 and $|\xi| < 1$, so the scheme is stable. However, if $4\alpha > 2$, ξ can become < -1 for some values of k and instability sets in. The stability condition then is $\alpha \le \frac{1}{2}$, or

$$\Delta t \le \frac{\Delta x^2}{2D},$$

which we can identify as the time needed for information to diffuse across one grid cell. For the case discussed in Exercise 9.1, $D = 1$ and $\Delta x = 0.1$, so the critical $\Delta t = 0.005$, as observed.