## Von Neumann Stability Analysis

In Exercise 6.1 we saw that our simple explicit FTCS differencing of the advection equation is unstable. Why should that be? A simple but powerful analytical tool developed by computational physicist John von Neumann in the 1950s tells us why.

The basic differencing scheme is

$$u_j^{n+1} = u_j^n - \alpha\left(u_{j+1}^n - u_{j-1}^n\right), \tag{1}$$

where $\alpha = v\Delta t/2\Delta x$. This is a *linear* transformation from the old state of the system $\mathbf{u}^n$ to the new state $\mathbf{u}^{n+1}$, where the vector $\mathbf{u} = (u_0, u_1, \ldots, u_{J-1})$. As with any linear operator, the eigenvectors of the system are of great interest, so let's first decompose the field $\mathbf{u}$ on the grid into Fourier components $e^{ikx}$, and then look at the behavior of each component under the transformation described in Equation 1. [The continuum version of this basic approach is the starting point for almost all stability analyses of linear systems in physics.]

In our discretization scheme, $x_j = x_0 + j\Delta x$ so, ignoring a phase factor, a pure Fourier mode with wavenumber $k = 2\pi/\lambda$ has $e^{ikx_j} = e^{ikj\Delta x}$ on the grid points. Since $k$ is unchanged by a linear transformation, we expect that after a single application of the differencing scheme, this mode will simply be modified in amplitude and phase:

$$e^{ikj\Delta x} \;\to\; \xi(k)e^{ikj\Delta x}.$$

Here, $\xi(k)$ is the (complex) amplification factor for a mode of wavenumber $k$. If $|\xi| > 1$ the mode grows in amplitude with each iteration and the method is unstable. If any mode has $|\xi| > 1$, the system is unstable—some component with that particular $k$ will eventually explode. Each time step corresponds to one application of the transformation, so if the initial state looks like $e^{ikx_j}$

$$u_j^0 = e^{ikj\Delta x},$$

the $n$-th state has

$$u_j^n = \xi(k)^n\, e^{ikj\Delta x}.$$

Note by the way that the superscript $n$ on $u$ means the $n$-th time step, while on $\xi$ it means a power. Sorry—we're following Numerical Recipes in this development...

Substituting this expression for $u_j^n$ into Equation 1 allows us to determine which modes are stable. We find

$$\xi^{n+1}\, e^{ikj\Delta x} = \xi^n\, e^{ikj\Delta x} - \alpha\left(\xi^n\, e^{ik(j+1)\Delta x} - \xi^n\, e^{ik(j-1)\Delta x}\right).$$

Dividing through by $\xi^n\, e^{ikj\Delta x}$, this simplifies to

$$\begin{aligned}\xi(k) &= 1 - \alpha\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\\ &= 1 - 2i\alpha\sin k\Delta x.\end{aligned} \tag{2}$$

Thus, $|\xi(k)| > 1$ for <u>all</u> $k$, as observed!

It is also worth noting that the above expression for $\xi(k)$ also tells us *which* modes are expected to go unstable first as $\alpha$ increases. Clearly the first mode to reach $\xi = -1$ will be the one for which $\sin\frac{1}{2}k\Delta x = 1$, or $k\Delta x = \pi$. This corresponds to a wavelength $\lambda = 2\pi/k = 2\Delta x$, i.e. the grid scale, again as observed.

The von Neumann stability analysis is a very important technique that gives us important analytical insight into the behavior of numerical integration schemes — even if they are not linear.