Mean and variance of a random walk

A random walk $\{x^n\}$ is the sum of a series of random steps:

$$x_n = \sum_{i=0}^{n-1} \delta x_i,$$

where the steps δx_i are uncorrelated and drawn from the same probability distribution, independent of i. For the symmetric walk considered here, the distribution is simply that $\delta x = \pm\Delta x$, with probability $\frac{1}{2}$ for each outcome.

We can easily calculate the expectation value and variance of x_n:

$$\begin{aligned}
\langle x_n \rangle &= \left\langle \sum_{i=0}^{n-1} \delta x_i \right\rangle \\
&= \sum_{i=0}^{n-1} \langle \delta x_i \rangle \\
&= n \langle \delta x \rangle \\
&= 0 \quad \text{here.}
\end{aligned}$$

$$\begin{aligned}
\langle x_n^2 \rangle &= \left\langle \sum_{i=0}^{n-1} \delta x_i \sum_{j=0}^{n-1} \delta x_j \right\rangle \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \langle \delta x_i \delta x_j \rangle \\
&= n\,\Delta x^2,
\end{aligned}$$

where we have used the fact that $\langle \delta x_i \delta x_j \rangle = \Delta x^2 \delta_{ij}$, since the steps are uncorrelated.

Thus, if we average over an ensemble of walkers, the distribution should have zero mean and will spread out with rms width proportional to $\sqrt{n}$. In fact, it can be shown that, as the number of walkers increases, the distribution is gaussian.