## Random walks and diffusion

A random walk $\{x^n\}$ is the sum of a series of random steps:

$$x_n = \sum_{i=0}^{n-1} \delta x_i,$$

where the steps $\delta x_i$ are uncorrelated and drawn from the same probability distribution, independent of $i$.

Let's regard the number of steps $n$ as a time variable, with $t_n = n\Delta t$. Then, again considering an ensemble of many walkers, we can define $N_i(t)$ as the number of walkers currently in state $i$, and the probability that a walker moves from state $i$ to state $i \pm 1$ in time $\Delta t$ is $R_\pm \Delta t$. Then the change in $N_i$ is the number of walkers moving into that state minus the number of walkers moving out:

$$N_i(t+\Delta t) = N_i(t) - R_-\Delta t N_i(t) - R_+\Delta t N_i(t) + R_+\Delta t N_{i-1}(t) + R_-\Delta t N_{i+1}(t).$$

Rearranging and setting $R_\pm = R$, this implies

$$\begin{aligned}\frac{N_i(t+\Delta t) - N_i(t)}{\Delta t} &= R\left[N_{i+1}(t) - 2N_i(t) + N_{i-1}(t)\right] \\ &= R\Delta x^2 \, \frac{N_{i+1}(t) - 2N_i(t) + N_{i-1}(t)}{\Delta x^2},\end{aligned}$$

which we recognize as the FTCS differenced form of the diffusion equation

$$\frac{\partial N}{\partial t} = D\frac{\partial^2 N}{\partial x^2},$$

where we can now identify the diffusion coefficient as

$$D = R\Delta x^2.$$