Fields Due to Moving Charges

The electric field at position $\mathbf{r}$ due to a stationary charge q at location $\mathbf{R}$ may be written as

$$\mathbf{E} = -\frac{\partial \phi}{\partial \mathbf{r}}\,,$$

where the potential is

$$\phi(\mathbf{r}) = \frac{kq}{|\mathbf{r} - \mathbf{R}|}$$

and the derivative denotes the gradient with respect to the field point $\mathbf{r}$.

For a *moving* charge, this expression must be modified in two important ways. First, because electromagnetic information travels at a finite speed—the speed of light, c—the relevant $\mathbf{R}$ to use is the position of the source charge at the so-called *retarded time* $t_{\rm ret}$, defined implicitly by

$$\begin{aligned} t_{\rm ret} &= t - r_{\rm ret}/c\,, \\ r_{\rm ret} &= |\mathbf{r}_{\rm ret}| = |\mathbf{r} - \mathbf{R}(t_{\rm ret})|\,. \end{aligned}$$

This is the time at which the signal received at position $\mathbf{r}$ at time t was emitted by the source. Second, a relativistic "Doppler" factor must be included to take into account the relative motion of source and observer. The final expression for the potential then is

$$\phi(\mathbf{r}, t) = \frac{kq}{r_{\rm ret}\,(1 - \hat{\mathbf{r}}_{\rm ret}\cdot \mathbf{V}_{\rm ret}/c)}\,,$$

where

$$\mathbf{V}_{\rm ret} = \left.\frac{d\mathbf{R}(t)}{dt}\right|_{t=t_{\rm ret}}$$

and $\hat{\mathbf{r}}_{\rm ret}$ is the unit vector in the direction $\mathbf{r}_{\rm ret}$.

Similar considerations apply to the vector potential $\boldsymbol{\mathcal{A}}$ which determines the magnetic field:

$$\mathbf{B} = \nabla \times \boldsymbol{\mathcal{A}}\,.$$

For a source charge q at location $\mathbf{R}$ moving with nonrelativistic velocity $\mathbf{V}$, the vector potential at $\mathbf{r}$ is

$$\boldsymbol{\mathcal{A}} = \frac{kq}{|\mathbf{r} - \mathbf{R}|}\,\frac{\mathbf{V}}{c}\,.$$

The relativistic version of this equation is

$$\boldsymbol{\mathcal{A}}(\mathbf{r}, t) = \frac{kq\,\mathbf{V}_{\rm ret}/c}{r_{\rm ret}\,(1 - \hat{\mathbf{r}}_{\rm ret}\cdot \mathbf{V}_{\rm ret}/c)}\,.$$

In terms of ϕ and $\boldsymbol{\mathcal{A}}$, the electric field is

$$\mathbf{E} = -\frac{\partial \phi}{\partial \mathbf{r}} - \frac{1}{c}\frac{\partial \boldsymbol{\mathcal{A}}}{\partial t}\,.$$

We can expand this expression to obtain

$$\mathbf{E}(\mathbf{r}, t) = \frac{kq\,r_{\rm ret}}{(\mathbf{r}_{\rm ret}\cdot \mathbf{u}_{\rm ret})^3}\Big[\mathbf{u}_{\rm ret}(c^2 - V_{\rm ret}^2) + \mathbf{r}_{\rm ret} \times (\mathbf{u}_{\rm ret} \times \mathbf{A}_{\rm ret})\Big]$$

and

$$\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{r}}_{\rm ret} \times \mathbf{E}(\mathbf{r},t)\,,$$

where

$$\mathbf{u}_{\rm ret} = c\,\hat{\mathbf{r}}_{\rm ret} - \mathbf{V}_{\rm ret}$$

and

$$\mathbf{A}_{\rm ret} = \left.\frac{d\mathbf{V}(t)}{dt}\right|_{t=t_{\rm ret}}\,.$$

That's basically all there is to know about relativistic electrodynamics!

From a practical standpoint, the main problem inherent in the above discussion is that the implicit equation for $t_{\rm ret}$ is in general insoluble by analytic means, so we must resort to numerical techniques. Specifically, once the function $\mathbf{R}(t)$ is specified, to find $t_{\rm ret}$ we must solve

$$f(t_{\rm ret};t,\mathbf{r}) \equiv t - t_{\rm ret} - r_{\rm ret}(t_{\rm ret})/c = 0$$

(where t and $\mathbf{r}$ are simply fixed parameters, at least as far as actually solving the equation is concerned). The explicit form is

$$t - t_{\rm ret} - \frac{1}{c}\sqrt{[\mathbf{r} - \mathbf{R}(t_{\rm ret})]^2} = 0\,.$$

It must be solved *every* time the field is recomputed.

A variety of root-finding techniques are discussed elsewhere in the Computational Physics pages. For our purposes, the bisection method should be adequate. Once $t_{\rm ret}$ is known, we can proceed to calculate and use the fields as usual.