PHYS 325: Computational Physics III

Winter 2023

Homework #4
(Due: February 24, 2023)

1. Write a program to find the numerical solution $u(x,t)$ of the 1-dimensional advection equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0\,,$$

for $-10 \le x \le 10$, subject to the initial condition

$$u(x,0) = \quad e^{-2(x-2)^2}$$

using the *Lax-Wendroff* method:

$$u_j^{n+1} = u_j^n - \alpha\left[\tfrac{1}{2}(u_{j+1}^n + u_j^n) - \tfrac{1}{2}\alpha(u_{j+1}^n - u_j^n) - \tfrac{1}{2}(u_j^n + u_{j-1}^n) + \tfrac{1}{2}\alpha(u_j^n - u_{j-1}^n)\right]\,,$$

where $\alpha = v\Delta t/\Delta x$. Perform the calculation with $v = 1$ on a grid of 501 points spanning the range $-10 \le x \le 10$ (so $\Delta x = 0.04$), with time step $\Delta t = \Delta x$, and take $v = 1$.

(a) How does your numerical solution compare with the analytical solution at times $t = 1, 2, 5,$ and 10? Plot the numerical and analytical solutions at each time on the same graph. Finally, plot the maximum absolute difference between the numerical and analytic solutions as a function of time for $0 \le t \le 10$.

(b) Repeat part (a) with a time step of $0.5\Delta x$.

2. As discussed in class, the 1-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0\,,$$

can be recast in a form similar to the advection equation by setting

$$\begin{aligned} r &= c\frac{\partial u}{\partial x} \\ s &= \frac{\partial u}{\partial t} \end{aligned}$$

to obtain

$$\begin{aligned} \frac{\partial r}{\partial t} &= c\frac{\partial s}{\partial x} \\ \frac{\partial s}{\partial t} &= c\frac{\partial r}{\partial x} \end{aligned}$$

(a) Derive (or look up in Numerical Recipes) the form of the Lax scheme for this problem, and write down the rule for advancing the system from time n to time $n + 1$.

(b) Hence write a program to solve the wave equation on the same grid as in question 1, with initial conditions

$$
\begin{aligned}
u(x,0) &= \sin(\pi x) && -1 \le x \le 1 \\
u(x,0) &= 0 && \text{otherwise} \\
\\
\partial u/\partial t &= 0 && \text{for all } x
\end{aligned}
$$

Choose $c = 1$ and $\Delta t = \Delta x$, and plot the numerical and analytical solutions for r and u at time $t = 5$ on the same graph. Turn in both your program and the resulting graph.

3. Write a program to solve the *two*-dimensional version of problem 1 using the Lax scheme, to determine the solution $u(x, y, t)$ of the equation

$$\frac{\partial u}{\partial t} + (\mathbf{v} \cdot \nabla)u = 0\,,$$

where $\mathbf{v} = (v_x, v_y)$ is a constant flow vector. Work on a square 201×201 grid in x and y with $-10 \le x, y \le 10$ and $\Delta x = \Delta y = \Delta = 0.1$. Take $u(x, y, 0) = \sin \pi r'$ for $r' < 2$ and 0 otherwise, where $r' = \sqrt{(x + 3)^2 + y^2}$, and let $v_x = \frac{1}{2}\sqrt{3}$, $v_y = 0.5$. Use a time step $\Delta t = \Delta/\sqrt{2}$. Turn in images of the $u(x, y)$ field at times $t = 0, 1, 2, 5$ and 10. What happens if you set $\Delta t = \Delta$, as in the 1-D case?