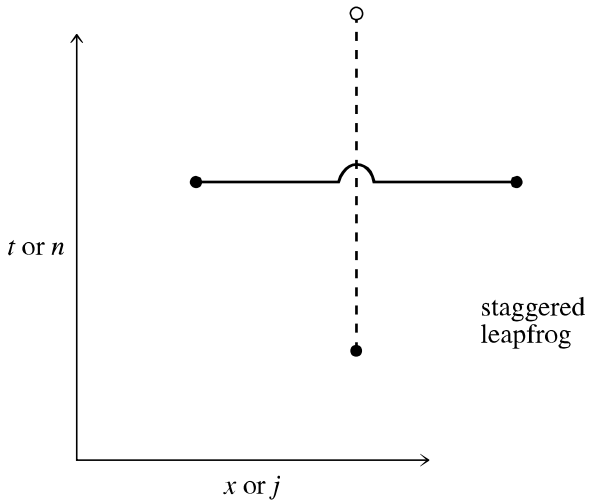

Figure 19.1.5. Representation of the staggered leapfrog differencing scheme. Note that information from two previous time slices is used in obtaining the desired point. This scheme is second-order accurate in both space and time.

arrives at a given point after a time interval Δt. In the first-order method, the material always arrives from Δx away. If $v\Delta t \ll \Delta x$ (to insure accuracy), this can cause a large error. One way of reducing this error is to interpolate u between $j-1$ and j before transporting it. This gives effectively a second-order method. Various schemes for second-order upwind differencing are discussed and compared in [2-3].

Second-Order Accuracy in Time

When using a method that is first-order accurate in time but second-order accurate in space, one generally has to take $v\Delta t$ significantly smaller than Δx to achieve desired accuracy, say, by at least a factor of 5. Thus the Courant condition is not actually the limiting factor with such schemes in practice. However, there are schemes that are second-order accurate in both space and time, and these can often be pushed right to their stability limit, with correspondingly smaller computation times.

For example, the *staggered leapfrog* method for the conservation equation (19.1.1) is defined as follows (Figure 19.1.5): Using the values of u^n at time t^n, compute the fluxes F_j^n. Then compute new values u^{n+1} using the time-centered values of the fluxes:

$$u_j^{n+1} - u_j^{n-1} = -\frac{\Delta t}{\Delta x}(F_{j+1}^n - F_{j-1}^n) \tag{19.1.30}$$

The name comes from the fact that the time levels in the time derivative term "leapfrog" over the time levels in the space derivative term. The method requires that u^{n-1} and u^n be stored to compute u^{n+1}.

For our simple model equation (19.1.6), staggered leapfrog takes the form

$$u_j^{n+1} - u_j^{n-1} = -\frac{v\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n) \tag{19.1.31}$$

The von Neumann stability analysis now gives a quadratic equation for ξ, rather than a linear one, because of the occurrence of three consecutive powers of ξ when the

form (19.1.12) for an eigenmode is substituted into equation (19.1.31),

$$\xi^2 - 1 = -2i\xi \frac{v\Delta t}{\Delta x} \sin k\Delta x \tag{19.1.32}$$

whose solution is

$$\xi = -i\frac{v\Delta t}{\Delta x}\sin k\Delta x \pm \sqrt{1 - \left(\frac{v\Delta t}{\Delta x}\sin k\Delta x\right)^2} \tag{19.1.33}$$

Thus the Courant condition is again required for stability. In fact, in equation (19.1.33), $|\xi|^2 = 1$ for any $v\Delta t \le \Delta x$. This is the great advantage of the staggered leapfrog method: There is no amplitude dissipation.

Staggered leapfrog differencing of equations like (19.1.20) is most transparent if the variables are centered on appropriate half-mesh points:

$$\begin{aligned} r^n_{j+1/2} &\equiv v\left.\frac{\partial u}{\partial x}\right|^n_{j+1/2} = v\frac{u^n_{j+1} - u^n_j}{\Delta x} \\ s^{n+1/2}_j &\equiv \left.\frac{\partial u}{\partial t}\right|^{n+1/2}_j = \frac{u^{n+1}_j - u^n_j}{\Delta t} \end{aligned} \tag{19.1.34}$$

This is purely a notational convenience: we can think of the mesh on which r and s are defined as being twice as fine as the mesh on which the original variable u is defined. The leapfrog differencing of equation (19.1.20) is

$$\begin{aligned} \frac{r^{n+1}_{j+1/2} - r^n_{j+1/2}}{\Delta t} &= \frac{s^{n+1/2}_{j+1} - s^{n+1/2}_j}{\Delta x} \\ \frac{s^{n+1/2}_j - s^{n-1/2}_j}{\Delta t} &= v\frac{r^n_{j+1/2} - r^n_{j-1/2}}{\Delta x} \end{aligned} \tag{19.1.35}$$

If you substitute equation (19.1.22) in equation (19.1.35), you will find that once again the Courant condition is required for stability, and that there is no amplitude dissipation when it is satisfied.

If we substitute equation (19.1.34) in equation (19.1.35), we find that equation (19.1.35) is equivalent to

$$\frac{u^{n+1}_j - 2u^n_j + u^{n-1}_j}{(\Delta t)^2} = v^2\frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{(\Delta x)^2} \tag{19.1.36}$$

This is just the "usual" second-order differencing of the wave equation (19.1.2). We see that it is a two-level scheme, requiring both u^n and u^{n-1} to obtain u^{n+1}. In equation (19.1.35) this shows up as both $s^{n-1/2}$ and r^n being needed to advance the solution.

For equations more complicated than our simple model equation, especially nonlinear equations, the leapfrog method usually becomes unstable when the gradients get large. The instability is related to the fact that odd and even mesh points are completely decoupled, like the black and white squares of a chess board, as shown

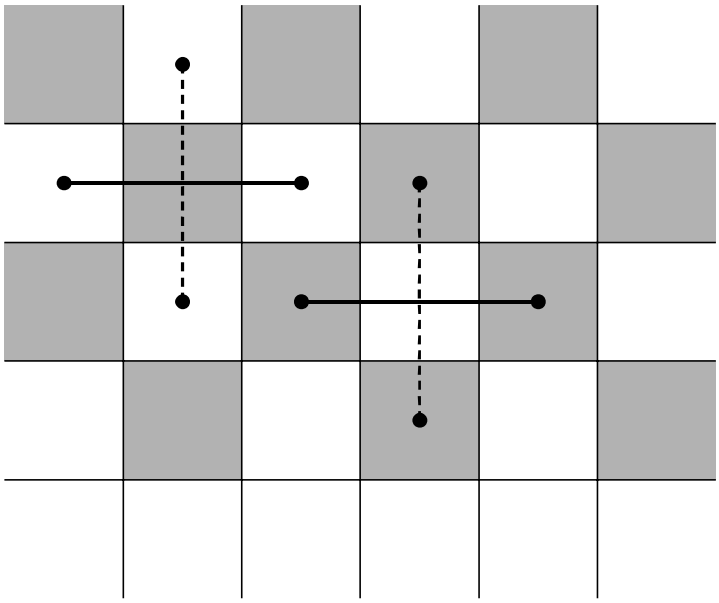

Figure 19.1.6. Origin of mesh-drift instabilities in a staggered leapfrog scheme. If the mesh points are imagined to lie in the squares of a chess board, then white squares couple to themselves, black to themselves, but there is no coupling between white and black. The fix is to introduce a small diffusive mesh-coupling piece.

in Figure 19.1.6. This mesh drifting instability is cured by coupling the two meshes through a numerical viscosity term, e.g., adding to the right side of (19.1.31) a small coefficient ($\ll 1$) times $u^n_{j+1} - 2u^n_j + u^n_{j-1}$. For more on stabilizing difference schemes by adding numerical dissipation, see, e.g., [4].

The *Two-Step Lax-Wendroff* scheme is a second-order in time method that avoids large numerical dissipation and mesh drifting. One defines intermediate values $u_{j+1/2}$ at the half timesteps $t_{n+1/2}$ and the half mesh points $x_{j+1/2}$. These are calculated by the Lax scheme:

$$u^{n+1/2}_{j+1/2} = \frac{1}{2}(u^n_{j+1} + u^n_j) - \frac{\Delta t}{2\Delta x}(F^n_{j+1} - F^n_j) \tag{19.1.37}$$

Using these variables, one calculates the fluxes $F^{n+1/2}_{j+1/2}$. Then the updated values u^{n+1}_j are calculated by the properly centered expression

$$u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta x}\left(F^{n+1/2}_{j+1/2} - F^{n+1/2}_{j-1/2}\right) \tag{19.1.38}$$

The provisional values $u^{n+1/2}_{j+1/2}$ are now discarded. (See Figure 19.1.7.)

Let us investigate the stability of this method for our model advective equation, where $F = vu$. Substitute (19.1.37) in (19.1.38) to get

$$\begin{aligned} u^{n+1}_j = u^n_j - \alpha \Bigg[& \frac{1}{2}(u^n_{j+1} + u^n_j) - \frac{1}{2}\alpha(u^n_{j+1} - u^n_j) \\ & - \frac{1}{2}(u^n_j + u^n_{j-1}) + \frac{1}{2}\alpha(u^n_j - u^n_{j-1}) \Bigg] \end{aligned} \tag{19.1.39}$$

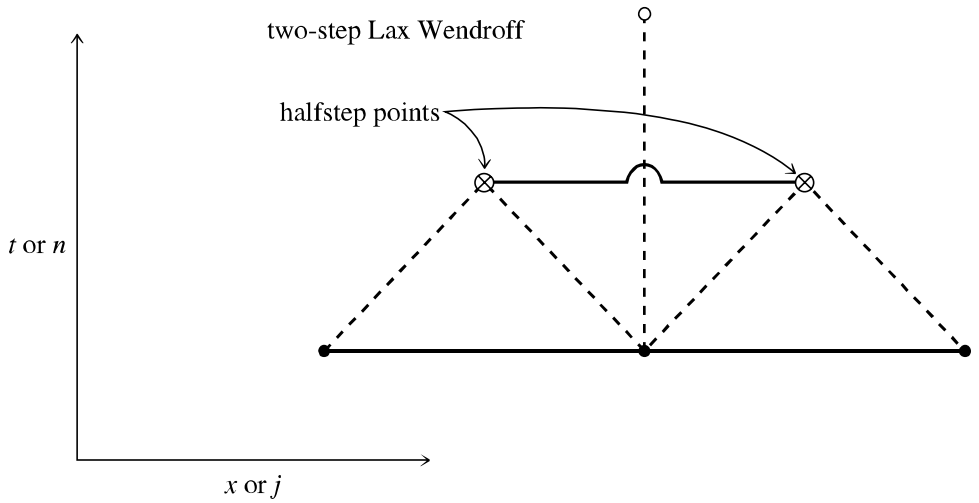

Figure 19.1.7. Representation of the two-step Lax-Wendroff differencing scheme. Two halfstep points (⊗) are calculated by the Lax method. These, plus one of the original points, produce the new point via staggered leapfrog. Halfstep points are used only temporarily and do not require storage allocation on the grid. This scheme is second-order accurate in both space and time.

where

$$\alpha \equiv \frac{v\Delta t}{\Delta x} \tag{19.1.40}$$

Then

$$\xi = 1 - i\alpha \sin k\Delta x - \alpha^2(1 - \cos k\Delta x) \tag{19.1.41}$$

so

$$|\xi|^2 = 1 - \alpha^2(1 - \alpha^2)(1 - \cos k\Delta x)^2 \tag{19.1.42}$$

The stability criterion $|\xi|^2 \leq 1$ is therefore $\alpha^2 \leq 1$, or $v\Delta t \leq \Delta x$ as usual. Incidentally, you should not think that the Courant condition is the only stability requirement that ever turns up in PDEs. It keeps doing so in our model examples just because those examples are so simple in form. The method of analysis is, however, general.

Except when $\alpha = 1$, $|\xi|^2 < 1$ in (19.1.42), so some amplitude damping does occur. The effect is relatively small, however, for wavelengths large compared with the mesh size Δx. If we expand (19.1.42) for small $k\Delta x$, we find

$$|\xi|^2 = 1 - \alpha^2(1 - \alpha^2)\frac{(k\Delta x)^4}{4} + \dots \tag{19.1.43}$$

The departure from unity occurs only at fourth order in k. This should be contrasted with equation (19.1.16) for the Lax method, which shows that

$$|\xi|^2 = 1 - (1 - \alpha^2)(k\Delta x)^2 + \dots \tag{19.1.44}$$

for small $k\Delta x$.