# Computation of Electric Field Lines I

## Field Lines and Equipotentials

For definiteness and simplicity, let's begin our study by computing the electric field lines due to a group of stationary point charges. Let the $n_q$ charges be $q_i$ ($i = 1, \ldots, n_q$), with positions $\mathbf{x}_i$. Again for simplicity and ease of visualization, we will work in two dimensions, writing $\mathbf{x}_i = (x_i, y_i)$. The electric potential at some arbitrary point $\mathbf{x} = (x, y)$ is then

$$\phi(\mathbf{x}) = \sum_{i=1}^{n_q} \frac{kq_i}{|\mathbf{x} - \mathbf{x}_i|} \, ,$$

where $|\mathbf{x} - \mathbf{x}_i| = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ and $k = 9.0 \times 10^9$ in SI units. The electric field is

$$\mathbf{E}(\mathbf{x}) = -\nabla\phi(\mathbf{x}) = \sum_{i=1}^{n_q} \frac{kq_i(\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^3} \, .$$

A *field line* is defined as a curve that is everywhere tangent to the local electric field vector. Mathematically, if we denote this curve by $\mathbf{X}(s)$, where $s$ is distance along the field line, the above definition means that the tangent vector $d\mathbf{X}/ds$ must be equal to the unit vector in the direction of $\mathbf{E}$, i.e.

$$\frac{d\mathbf{X}}{ds} = \hat{\mathbf{E}} \equiv \frac{\mathbf{E}}{|\mathbf{E}|} \, .$$

This definition will translate directly into the numerical method for computing the field line.

An *equipotential* is a surface (curve in 2-D) on which the electric potential $\phi$ is constant. It is easy to show that field lines are always perpendicular to equipotential surfaces, as follows. Suppose $\mathbf{X}$ and $\mathbf{X} + \delta\mathbf{X}$ lie on the same equipotential, so the vector $\delta\mathbf{X}$ is tangent to the surface. The potential difference between the two points is $\delta\phi = -\mathbf{E}.\delta\mathbf{X}$ (by definition of $\mathbf{E}$) $= 0$ (since the points are at the same potential). Therefore $\delta\mathbf{X}$ is orthogonal to $\mathbf{E}$. Hence, instead of

$$\frac{d\mathbf{X}}{ds} = \left( \frac{E_x}{|\mathbf{E}|}, \frac{E_y}{|\mathbf{E}|} \right)$$

as above, we now integrate

$$\frac{d\mathbf{X}}{ds} = \left( -\frac{E_y}{|\mathbf{E}|}, \frac{E_x}{|\mathbf{E}|} \right) .$$

Think of the equipotentials and field lines as forming an orthogonal "coordinate system" of sorts spanning the space containing the charges. Note that we could in principle use $\phi$ instead of $s$ as the parameter describing position along a field line, as $\phi$ decreases monotonically along the line. In fact, we will find that a combination of the two is most convenient.

## Operational Definition of a Field Line

The procedure for computing a field line is very straightforward, and (apart from the fact that $\delta s$ is length along the curve instead of a separate variable) is essentially the Euler method for solving ordinary differential equations. Indeed, we can improve the accuracy of the method by going to higher orders, as with the Runge-Kutta or predictor–corrector ODE solvers, as outlined below. Given a point $\mathbf{X}$ on some field line, we take a step of length $\delta s$ in the direction of the field:

$$\mathbf{X} \quad \longrightarrow \quad \mathbf{X} + \delta\mathbf{X} \quad = \quad \mathbf{X} + \delta s \left( \frac{\mathbf{E}}{|\mathbf{E}|} \right) .$$

In the limit $\delta s \to 0$, the new point will lie on the field line, according to the above differential equation. In practice, we choose $\delta s$ as small as is practical, and regard the result as an approximation to the desired line.

The simplest higher-order generalization of this method is, as usual, to try to use the direction of the field in the *middle* of the step to define $\delta\mathbf{X}$, instead of the field at the start. A straightforward way to do this is as follows. First let

$$\mathbf{X}' \quad = \quad \mathbf{X} + \delta s \left( \frac{\mathbf{E}}{|\mathbf{E}|} \right) ,$$

just as before. Then take a step using an estimate of the field at the mid-point:

$$\mathbf{X} \quad \longrightarrow \quad \mathbf{X} + \delta\mathbf{X} \quad = \quad \mathbf{X} + \delta s \left( \frac{\mathbf{E}_{\rm av}}{|\mathbf{E}_{\rm av}|} \right) ,$$

where $\mathbf{E}_{\rm av} = \frac{1}{2}[\mathbf{E}(\mathbf{X}) + \mathbf{E}(\mathbf{X}')]$. The similarities to the second-order schemes mentioned earlier should be obvious.

Given this procedure for computing the field line through a given point, mapping out the field line structure of our distribution of charges is easy. We know that field lines start on positive charges and either end on a negative charge or extend to infinity so, starting near each charge, we simply repeat the above step—moving parallel to the field away from a positive charge, and opposite to the field for a negative charge—until we reach another charge or the field line exceeds some specified distance from the origin.

The scripts `field_line_loop.py` and `field_line_loop_2.py` present basic functions to draw a field line starting from a given point. They can step either forward or backward, depending on the parameter `direction`, and stop whenever either of the two termination criteria just mentioned are satisfied. The first script isn't much use, as it doesn't actually return any information—it just illustrates the algorithm. The second version returns two lists containing the x- and y-coordinates of the field line.