PHYS 325: Computational Physics III

Winter 2023

Exercise 7.2

1. Modify the (provided) Lax integrator for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

to implement a Lax-Wendroff scheme. Written in flux-conservative form, the wave equation is (see Numerical Recipes)

$$\frac{\partial \mathbf{y}}{\partial t} = -\frac{\partial \mathbf{F}(\mathbf{y})}{\partial x},$$

where $\mathbf{y}$ is the vector

$$\mathbf{y} = \begin{pmatrix} r \\ s \end{pmatrix},$$

r and s are defined by

$$\begin{aligned} r &= v\frac{\partial u}{\partial x} \\ s &= \frac{\partial u}{\partial t}, \end{aligned}$$

and the flux vector $\mathbf{F}$ for this problem is

$$\mathbf{F}(\mathbf{y}) = \begin{pmatrix} 0 & -v \\ -v & 0 \end{pmatrix} \mathbf{y} = -v \begin{pmatrix} s \\ r \end{pmatrix}.$$

With this terminology, the Lax-Wendroff scheme can be written compactly as†

$$\begin{aligned} \mathbf{y}_{j-1/2}^{n+1/2} &= \tfrac{1}{2}(\mathbf{y}_j^n + \mathbf{y}_{j-1}^n) - \tfrac{1}{2}\alpha(\mathbf{F}_j^n - \mathbf{F}_{j-1}^n) \\ \mathbf{y}_{j+1/2}^{n+1/2} &= \tfrac{1}{2}(\mathbf{y}_j^n + \mathbf{y}_{j+1}^n) - \tfrac{1}{2}\alpha(\mathbf{F}_{j+1}^n - \mathbf{F}_j^n) \\ \mathbf{y}_j^{n+1} &= \mathbf{y}_j^n - \alpha(\mathbf{F}_{j+1/2}^{n+1/2} - \mathbf{F}_{j-1/2}^{n+1/2}), \end{aligned}$$

where $\alpha = v\Delta t/\Delta x$ and we define

$$\mathbf{F}_j^n = \mathbf{F}(\mathbf{y}_j^n) = -v \begin{pmatrix} s_j^n \\ r_j^n \end{pmatrix}.$$

We will take $v = 1$. Note that, as in the Lax implementation, you will have to integrate the s equation as you go to obtain the field u. For initial conditions, use

$$\begin{aligned} u(x,0) &= e^{-(x+7)^2} \\ r(x,0) &= v\frac{\partial u}{\partial x} \\ s(x,0) &= -r(x,0). \end{aligned}$$

Again determine the speed at which the disturbance moves across the grid.

†The flux-conservative formulation is a very general way to write down the problem, and should be the approach you use if you can, but in case you are feeling overwhelmed by the many components and indices, here is the same scheme written out explicitly in terms of r and s.

$$\begin{aligned}
r_{j-1/2}^{n+1/2} &= \tfrac{1}{2}(r_j^n + r_{j-1}^n) + \tfrac{1}{2}\alpha(s_j^n - s_{j-1}^n) \\
s_{j-1/2}^{n+1/2} &= \tfrac{1}{2}(s_j^n + s_{j-1}^n) + \tfrac{1}{2}\alpha(r_j^n - r_{j-1}^n) \\
r_{j+1/2}^{n+1/2} &= \tfrac{1}{2}(r_j^n + r_{j+1}^n) + \tfrac{1}{2}\alpha(s_{j+1}^n - s_j^n) \\
s_{j+1/2}^{n+1/2} &= \tfrac{1}{2}(s_j^n + s_{j+1}^n) + \tfrac{1}{2}\alpha(r_{j+1}^n - r_j^n)
\end{aligned}$$

$$\begin{aligned}
r_j^{n+1} &= r_j^n + \alpha(s_{j+1/2}^{n+1/2} - s_{j-1/2}^{n+1/2}) \\
s_j^{n+1} &= s_j^n + \alpha(r_{j+1/2}^{n+1/2} - r_{j-1/2}^{n+1/2}).
\end{aligned}$$