Scattering by a Barrier

Consider a beam of particles incident from the left on a potential barrier localized near the origin. How many of those particles are reflected from the barriew, and how many are transmitted?

Suppose the barrier is represented by the potential

$$V(x) = \begin{cases} 0, & x < 0\,, \\ V_0, & 0 < x < a\,, \\ 0, & x > a\,, \end{cases}$$

where V_0 may be positive or negative. We can look for plane-wave solutions to the Schrödinger equation by assuming energy E and time-dependence

$$\psi(x,t) = \psi(x)\, e^{-iEt/\hbar}\,,$$

so that, in the regions where $V = 0$, the free-particle solutions to the time-independent equation are

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < 0 \\ Ce^{ikx}, & x > a \end{cases}$$

where $k^2 = 2mE/\hbar^2$.

The standard interpretation is that $|A|^2$ represents the amplitude (intensity) of the incident beam, $|B|^2$ the amplitude of the reflected beam, and $|C|^2$ the amplitude of the transmitted beam. In that case, we expect $|A|^2 = |B|^2 + |C|^2$ (conservation of mass, or probability), and the reflection and transmission probabilities are $R = |B/A|^2$ and $T = |C/A|^2$, respectively. These are the coefficients describing the properties of a 1-dimensional scattering potential.

Solving the Schrödinger equation is straightforward. We write down the general solution inside the well (different forms for $E < V_0$ and $E > V_0$, note), then match ψ and ψ' at $x = 0$ and $x = a$, and solve for B/A and C/A. Specifically, for $E > V_0$, the solution for $0 < x < a$ is

$$\psi = Fe^{i\alpha x} + Ge^{-i\alpha x},$$

where $\alpha^2 = 2m(E - V_0)/\hbar^2$. The continuity conditions are

$$\begin{aligned} A + B &= F + G \\ k(A - B) &= \alpha\,(F - G) \\ Fe^{i\alpha a} + Ge^{-i\alpha a} &= Ce^{ika} \\ \alpha\,(Fe^{i\alpha a} - Ge^{-i\alpha a}) &= Cke^{ika}. \end{aligned}$$

For $E < V_0$, the corresponding expressions are

$$\psi = Fe^{\beta x} + Ge^{-\beta x},$$

where $\beta^2 = 2m(V_0 - E)/\hbar^2$, and

$$\begin{aligned} A + B &= F + G \\ ik(A - B) &= \beta\,(F - G) \\ Fe^{\beta a} + Ge^{\beta a} &= Ce^{ika} \\ \beta\,(Fe^{\beta a} - Ge^{\beta a}) &= iCke^{ika}. \end{aligned}$$

Eliminating F and G in either case, we find

$$T = 1 - R = \begin{cases} \left[1 + \frac{V_0^2 \sin^2 \alpha a}{4E(E-V_0)}\right]^{-1} & (E > V_0) \\ \left[1 + \frac{V_0^2 \sinh^2 \beta a}{4E(V_0-E)}\right]^{-1} & (E < V_0) \end{cases}$$

These are the probabilities we will attempt to reproduce using the time-dependent Schrödinger equation applied to individual particle wave packets.