The Lax Method in Two Dimensions

Most of the ideas we saw in studying the advection equation in one dimension carry over to 2-D. Here are some details. We are now working with a 2-D array, $u^n_{jk} = u(x_j, y_k, t^n)$, with a 2-D set of initial conditions and boundary conditions on all four sides of the grid (just set the field u to zero on all sides, for now).

The equation to solve is

$$\frac{\partial u}{\partial t} = -(\mathbf{v} \cdot \nabla)u = -v_x \frac{\partial u}{\partial x} - v_y \frac{\partial u}{\partial y}.$$

We begin as in 1-D by writing down the FTCS formulation:

$$\begin{aligned}
\frac{\partial u}{\partial t} &\approx \frac{u^{n+1}_{jk} - u^n_{jk}}{\Delta t} \\
\frac{\partial u}{\partial x} &\approx \frac{u^n_{j+1,k} - u^n_{j-1,k}}{2\Delta} \\
\frac{\partial u}{\partial y} &\approx \frac{u^n_{j,k+1} - u^n_{j,k-1}}{2\Delta},
\end{aligned}$$

where we have taken $\Delta x = \Delta y = \Delta$. Substituting into the differential equation and rearranging, we find

$$u^{n+1}_{jk} = u^n_{jk} - \frac{v_x \Delta t}{2\Delta}\left(u^n_{j+1,k} - u^n_{j-1,k}\right) - \frac{v_y \Delta t}{2\Delta}\left(u^n_{j,k+1} - u^n_{j,k-1}\right).$$

Very simple—and completely unstable, for all the same reasons as before.

The fix, once again, is replacing the leading term on the right side with an average over its (now four) neighbors:

$$u^{n+1}_{jk} = \tfrac{1}{4}\left(u^n_{j+1,k} + u^n_{j-1,k} + u^n_{j,k+1} + u^n_{j,k-1}\right) - \frac{v_x \Delta t}{2\Delta}\left(u^n_{j+1,k} - u^n_{j-1,k}\right) - \frac{v_y \Delta t}{2\Delta}\left(u^n_{j,k+1} - u^n_{j,k-1}\right).$$