Simple Harmonic Motion

Simple harmonic motion is one of the most important problems in Physics. Huge areas of the field rest squarely on the properties of the classical and quantum harmonic oscillators. In-class Exercise 4.2 asks us to determine the motion of a classical harmonic system. Here we work through the steps needed to write down the analytic (exact) solution and relate it to the initial conditions of the dynamical problem.

The harmonic oscillator is a dynamical system in which the restoring force F tending to push a particle back toward the origin is propostional to the displacement x: $F = -Kx$. If the particle mass is m and its acceleration is a we can write Newton's law of motion as

$$ma = F = -Kx.$$

Writing $k = K/m$, the equation of motion be may written as

$$\frac{d^2x}{dt^2} = -kx. \tag{1}$$

The general mathematical solution of this equation is easily shown to be

$$x(t) = A\sin\Omega t + B\cos\Omega t, \tag{2}$$

where $\Omega^2 = k$ and A and B are arbitrary constants. Even if you haven't seen how to solve differential equations, you can easily verify this by substituting Eq. (2) into Eq. (1):

$$\begin{aligned} x &= A\sin\Omega t + B\cos\Omega t \\ \frac{dx}{dt} &= A\Omega\cos\Omega t - B\Omega\sin\Omega t \\ \frac{d^2x}{dt^2} &= -A\Omega^2\sin\Omega t - B\Omega^2\cos\Omega t \\ &= -\Omega^2 x. \end{aligned}$$

The general solution contains two unknown parameters. Physically, they correspond to the initial conditions of the problem $x(0) = x_0$ and $v(0) = v_0$, and can be simply related to them. Setting $t = 0$ in Eq. (2) and its derivative, we find

$$\begin{aligned} x_0 &= B \\ v_0 &= A\Omega. \end{aligned}$$

Thus, in terms of the initial conditions, the general solution is

$$\begin{aligned} x(t) &= (v_0/\Omega)\sin\Omega t + x_0\cos\Omega t, \\ v(t) &= v_0\cos\Omega t - x_0\Omega\sin\Omega t. \end{aligned}$$

For the particular problem in the exercise, $x_0 = 0$ and $k = 4$, so $\Omega = 2$ and the analytic solution is

$$\begin{aligned} x(t) &= \tfrac{1}{2}v_0\sin 2t \\ v(t) &= v_0\cos 2t. \end{aligned}$$