Butcher Tableaux

The general form of a Runge-Kutta scheme (state $n \to n+1$) to solve the ODE

$$\frac{dy}{dx} = f(x, y)$$

is

$$\begin{aligned}
\delta y_0 &= \delta x\, f(x_n + a_0\delta x,\, y_n) \\
\delta y_1 &= \delta x\, f(x_n + a_1\delta x,\, y_n + b_{10}\delta y_0) \\
\delta y_2 &= \delta x\, f(x_n + a_2\delta x,\, y_n + b_{20}\delta y_0 + b_{21}\delta y_1) \\
&. \\
&. \\
&. \\
\delta y_{s-1} &= \delta x\, f\left(x_n + a_{s-1}\delta x,\, y_n + \sum_{i=0}^{s-2} b_{s-1,i}\,\delta y_i\right) \\
y_{n+1} &= y_n \;+\; \sum_{i=0}^{s-1} c_i\delta y_i
\end{aligned}$$

(see Numerical Recipes, Sec. 17.2). This is not quite the syntax used previously in class or in the book—the count starts at 0 to facilitate translation of the algorithm into Python. The method is defined by s, the number of stages, the offsets a_i and b_{ji}, and the weights c_i.

The whole scheme is often written compactly in the form of a *Butcher tableau:*

$$\begin{array}{c|cccc}
a_0 & b_{00} & b_{01} & \cdots & b_{0,s-1} \\
a_1 & b_{10} & b_{11} & \cdots & b_{1,s-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{s-1} & b_{s-1,0} & b_{s-1,1} & \cdots & b_{s-1,s-1} \\
\hline
 & c_0 & c_1 & \cdots & c_{s-1}
\end{array}$$

Butcher tableaux for some of the schemes we have explored follow.

Euler

$$\begin{array}{c|c}
0 & \\
\hline
 & 1
\end{array}$$

Implicit Euler

$$\begin{array}{c|c}
1 & 1 \\
\hline
 & 1
\end{array}$$

Midpoint

$$
\begin{array}{c|cc}
0 & & \\
\frac{1}{2} & \frac{1}{2} & \\
\hline
 & 0 & 1
\end{array}
$$

3/8 Rule

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{3} & \frac{1}{3} & & & \\
\frac{2}{3} & -\frac{1}{3} & 1 & & \\
1 & 1 & -1 & 1 & \\
\hline
 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8}
\end{array}
$$

Runge-Kutta 4

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

Note that blank spaces in the 2-D b array should be interpreted as zeros, and any nonzero element on or above the diagonal means that this stage in the algorithm is implicit.