PHYS T580: The Standard Model

Homework #6

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1. **(SMIN 6.2)** Suppose, contrary to our work in this chapter, that the photon had a very small mass, 10^{-4} eV. What would the effective range of the electromagnetic force be? Express your answer in meters. Approximately how light (in kilograms) would the photon need to be such that earth-scale magnetic fields would still be measurable?
Solution:
The interaction field is approximately given by

$$E_{\text{int}} \approx \frac{e^{-mr}}{4\pi r}$$

For a measuralble field $E \approxeq 1$ so with $M = 10^{-4}\text{eV} \approx 1.8 \times 10^{-40}kg$ we have

$$\frac{e^{-mr}}{4\pi r} \approxeq 1 \qquad\qquad \Rightarrow \qquad r \approx 0.0795m$$

The magnetic field of earth is $B = 25 \times 10^{-9}T$ for this to be measurable in earth scale $r \approx 6.4 \times 10^6 m$ we again solve for m in the equation

$$B_{\text{int}} \approx \frac{e^{-mr}}{4m\pi r}$$
$$25 \times 10^{-9} \approx \frac{e^{-m6.4\times 10^6}}{4m\pi 6.4 \times 10^6} \qquad \Rightarrow m \approx 1.9 \times 10^{-7}kg$$

The mass of photon has to be very low in order for this to be measured. □

2. **(SMIN 6.6)** In classical electrodynamics, radiation is propagated along the Poynting vector,

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B},$$

an ordinary 3-vector. Express the components of S^i in terms of components of $F^{\mu\nu}$ in as simplified form as possible.
Solution:
In index notation the cross products of two vector is

$$S^i = \varepsilon_{ijk} E^j B^k$$

Since the magnetic field and electric field components in terms of the Farady tensor elements are

$$F^{0i} = E^i \qquad F^{ij} = B^k$$

The Poynting vector becomes

$$S^i = \varepsilon_{ijk} F^{0j} F^{ij} \qquad \Rightarrow \qquad \boldsymbol{S} = \begin{pmatrix} F^{02}F^{12} - F^{03}F^{31} \\ F^{03}F^{23} - F^{01}F^{12} \\ F^{01}F^{31} - F^{02}F^{23} \end{pmatrix}$$

This is the required Poynting vector in terms of the components of Farady tensor. □

3. **(SMIN 6.9)** In developing the two polarization-states model for the photon we lied upon $U(1)$ gauge invariance, which in turn depends on a massless photon. We know that a spin-1 particle are supposed to have three spin states, but we claimed that the third state was swallowed by the Coulomb gauge condition. Lets' approach the question of three states by assuming that the photon does have mass and obeys lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A^\mu A_\mu$$

(a) Write the Euler-Lagrange equation for the massive photon field.
Solution:
Since by definition the Farady tensor is the antisymmetric tensor formed by various derivatives of the components of A^μ.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad\qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

The product term in the lagrangian is:

$$\begin{aligned}F^{\mu\nu}F_{\mu\nu} &= \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\\ &= \partial^\mu A^\nu \partial_\mu \partial_\nu - \partial^\mu A^\nu \partial_\nu A_\mu - \partial^\nu A^\mu \partial_\mu A_\nu + \partial^\nu A^\mu \partial_\nu A_\mu\\ &= 2\left(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu\right)\end{aligned}$$

Writing out the lagrangian in terms of these components we get

$$\mathcal{L} = -\frac{1}{2}\left(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu\right) + \frac{1}{2}M^2 A^\mu A_\mu$$

Thus the Euler-Lagrange equations become

$$\begin{aligned}\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) &= \frac{\partial\mathcal{L}}{\partial A_\nu}\\ -\frac{1}{2}\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) &= \frac{1}{2}M^2 A^\nu\\ -\partial_\mu F^{\mu\nu} &= M^2 A^\nu\end{aligned}$$

These are the required Euler-Lagrange equations. □

(b) Let the photon field take the form of a single plane wave:

$$A^\mu = \varepsilon^\mu e^{-ip\cdot x}.$$

Express the Euler-Lagrange equations as dot products of p and ε with themselves and with each other. Show that the transverse wave condition drops out of the dispersion relation regardless of whether the field has mass.
Solution:
For this field the Farady tensor becomes

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -ip_\mu \varepsilon^\nu e^{-ip\cdot x} + ip_\nu \varepsilon^\mu e^{-ip\cdot x} = -i\left(p_\mu \varepsilon^\nu - p_\nu \varepsilon^\mu\right)e^{-ip\cdot x}$$

So the Euler-Lagrange equations become

$$\begin{aligned}-\partial_\mu F^{\mu\nu} &= M^2 A^\nu\\ -\left[-i\left(p_\mu\varepsilon^\nu - p_\nu\varepsilon^\mu\right)(-ip^\mu)e^{-ip\cdot x}\right] &= M^2 e^\nu e^{-ip\cdot x}\\ \left(p^\mu p_\mu \varepsilon^\nu - p_\nu \varepsilon^\mu p^\mu\right) &= M^2\varepsilon^\nu\\ \left(p\cdot p\varepsilon^\nu - p_\nu \varepsilon\cdot p\right) &= M^2\varepsilon^\nu\end{aligned}$$

Regardless of the mass the coefficient of p_ν on the LHS must be 0 so the dot product $\varepsilon\cdot p = 0$. □

(c) What is the third possible polarization-state for a massive photon propagating in the z-direction?
Solution:
For this vector field, $p \cdot p = M^2$ and $\varepsilon \cdot p = 0$. For a particle moving in z direction with moentum p_z and Energy E the momentum 4-vector is $p^\mu = \begin{pmatrix} E & 0 & 0 & p_z \end{pmatrix}^T$. The linearly independent ε vector satisfying these relations apart from the ones given is

$$\varepsilon_3 = \begin{pmatrix} p_z \\ 0 \\ 0 \\ E \end{pmatrix} \qquad \text{as} \qquad \varepsilon_3 \cdot p = \begin{pmatrix} p_z \\ 0 \\ 0 \\ E \end{pmatrix} \cdot \begin{pmatrix} E \\ 0 \\ 0 \\ p_z \end{pmatrix} = p_z E - E p_z = 0$$

Since the inner product of ε with itself is $p^2 - E^2 = -M^2$, we could choose normalization factor i/M for ε. □

(d) What are the electric and magnetic fields of the massive photon field in this third polarization state? What happens to theose fields for $m = 0$?
Solution:
Now the Electric and magnetic fields are simply the components of Farady tensor

$$\begin{aligned} E^i &= F^{0i} = -i\left(p_0\varepsilon^i - p_i\varepsilon^0\right)e^{ip\cdot x} \\ E_x &= F^{01} = -i\left(p_0\varepsilon^1 - p_1\varepsilon^0\right)e^{ip\cdot x} = 0 \\ E_y &= F^{02} = -i\left(p_0\varepsilon^2 - p_2\varepsilon^0\right)e^{ip\cdot x} = 0 \\ E_z &= F^{03} = -i\left(p_0\varepsilon^3 - p_3\varepsilon^0\right)e^{ip\cdot x} = -i(E^2 - p^2)e^{-ip\cdot x} = -iM^2e^{-ip\cdot x} \end{aligned}$$

$$\begin{aligned} B_k &= F^{ij} = -i\left(p_i\varepsilon^j - p_i\varepsilon^j\right)e^{ip\cdot x} \\ B_x &= F^{23} = -i\left(p_2\varepsilon^3 - p_3\varepsilon^2\right)e^{ip\cdot x} = 0 \\ B_y &= F^{31} = -i\left(p_3\varepsilon^1 - p_1\varepsilon^4\right)e^{ip\cdot x} = 0 \\ B_z &= F^{12} = -i\left(p_1\varepsilon^2 - p_2\varepsilon^1\right)e^{ip\cdot x} = 0 \end{aligned}$$

So $\boldsymbol{E} = \left[-iM^2e^{-ipx}\right]\hat{\mathbf{z}}$ and $\boldsymbol{B} = 0$. If $M = 0$ then the Electric field vanishes as well, so both the fields vanish. □

4. **(SMIN 6.10)** Consider an electron in a spin state

$$\phi = \begin{pmatrix} a \\ b \end{pmatrix}$$

in a magnetic field B_0 oriented along the z-axis. We will calculate the *Larmor Frequency* by which the electron precesses.

(a) Turn the interaction Hamiltonian into a first order differential equation in time.
Solution:
The given interaction hamiltonian is

$$\hat{H}_{\text{int}} = -\frac{q_e B_0}{2m}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Writing the hamiltonian as $i\partial_0 = i\frac{\partial}{\partial t}$

$$i\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{q_e B_0}{2m}\begin{pmatrix} a \\ -b \end{pmatrix}$$

So the differential equations are

$$i\frac{\partial a}{\partial t} = -\frac{q_e B_0}{2m}a \qquad i\frac{\partial b}{\partial t} = \frac{q_e B_0}{2m}b$$

These are the required differential equations. □

(b) Solve the differential equation in part a. What is the Frequency of oscillation of the phase *difference* between teh two components?
Solution:
The solutions are

$$a = a_0 e^{\frac{iq_e B_0}{2m} t} \qquad b = b_0 e^{-\frac{iq_e B_0}{2m} t}$$

The phase difference is

$$\varphi = \left(\frac{iq_e B_0 t}{2m}\right) - \left(-\frac{iq_e B_0 t}{2m}\right) = \frac{iq_e B_0 t}{m}$$

The frequency of oscillation is

$$\frac{iq_e B_0}{m}$$

This is the required frequency. $\square$