PHYS T580: The Standard Model

Homework #5

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1. **(SMIN 5.1)** Evaluate

   (a) $\{\gamma^0, \gamma^0\}$

   Solution:

$$\{\gamma^0, \gamma^0\} = \gamma^0\gamma^0 + \gamma^0\gamma^0 = 2\gamma^0\gamma^0 = 2\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = 2\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 2I_{4\times 4}$$

   The final matrix is the 4×4 identity matrix □

   (b) $\gamma^2\gamma^0\gamma^2$

   Solution:

$$\gamma^2\gamma^0\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \gamma^0$$

   □

   (c) $[\gamma^1, \gamma^2]$

   Solution:

$$\begin{aligned}
[\gamma^1, \gamma^2] &= \gamma^1\gamma^2 - \gamma^2\gamma^1 \\
&= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} - \begin{pmatrix} -\sigma_2\sigma_1 & 0 \\ 0 & -\sigma_2\sigma_1 \end{pmatrix} \\
&= \begin{pmatrix} [\sigma_2, \sigma_1] & 0 \\ 0 & [\sigma_2, \sigma_1] \end{pmatrix} = \begin{pmatrix} -2i\sigma_3 & 0 \\ 0 & -2i\sigma_3 \end{pmatrix}
\end{aligned}$$

   □

2. **(SMIN 5.3a)** Compute the various traces of the combinations og γ-matrices explicitly

   (a) $\mathrm{Tr}\left(\gamma^0\gamma^0\right)$

   Solution:

$$\gamma^0\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I_{4\times 4} \Rightarrow \mathrm{Tr}\left(\gamma^0\gamma^0\right) = 4$$

   □

(b) $\text{Tr}\left(\gamma^1\gamma^1\right)$
Solution:

$$\gamma^1\gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} \Rightarrow \text{Tr}\left(\gamma^1\gamma^1\right) = -4$$

□

(c) $\text{Tr}\left(\gamma^1\gamma^0\right)$
Solution:

$$\gamma^1\gamma^0 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} \Rightarrow \text{Tr}\left(\gamma^0\gamma^1\right) = 0$$

□

3. **(SMIN 5.7)** In quantum field theory calculations, we will often find it useful to cmpute the products like

$$\left[\bar{u}(1)\gamma^\mu u(2)\right],$$

where 1 corresponds to spin, mass and 4−momentum of a particle state, and 2 corresponds to similar quantities for second particle. For particle 1. $m = m_1$; $\boldsymbol{p} = 0$, and $s = +1/2$ and for particle 2, $m = 0$; $\boldsymbol{p} = p_z\hat{\mathbf{k}}$ and $s = +1/2$

(a) Calculate the vector values of $[\bar{u}(1)\gamma^\mu u(2)]$ for the states listed.
Solution:
For particle 1 $m = m_1$, $p = 0 \Rightarrow E = \sqrt{p^2 + m_1^2} = m_1$ and for particle 2 $m = 0$, $|\boldsymbol{p}| = p_z \Rightarrow E = \sqrt{p^2 + m_1^2} = p$, And $[\bar{u}(1)\gamma^\mu u(2)] = u^\dagger(1)\gamma^0\gamma^\mu u(2)$ so we we have

$$u(1) = \frac{m_1}{\sqrt{E+0}}\begin{pmatrix} 1 \\ 0 \\ \frac{E}{m_1} \\ 0 \end{pmatrix} \qquad \Rightarrow u^\dagger(1) = \sqrt{m_1}\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$$

$$u(2) = \frac{m}{\sqrt{E+p}}\begin{pmatrix} 1 \\ 0 \\ \frac{E+p}{m} \\ 0 \end{pmatrix} = \begin{pmatrix} m/\sqrt{E+p} \\ 0 \\ \sqrt{E+p} \\ 0 \end{pmatrix} \qquad \lim_{m\to 0} u(2) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2E} \\ 0 \end{pmatrix}$$

Alos the various product of gamma matrices are

$$\gamma^0\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad \gamma^0\gamma^1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

$$\gamma^0\gamma^2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \qquad \gamma^0\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

Using these to calculate the vectors we get
the various components are

$$u^\dagger(1)\gamma^0\gamma^0 u(2) = \sqrt{2m_1E} \qquad u^\dagger(1)\gamma^0\gamma^2 u(2) = 0$$
$$u^\dagger(1)\gamma^0\gamma^3 u(2) = \sqrt{2m_1E} \qquad u^\dagger(1)\gamma^0\gamma^1 u(2) = 0$$

So the required matrix is

$$[\bar{u}(1)\gamma^\mu u(2)] = \begin{pmatrix} \sqrt{2m_1 E} \\ 0 \\ 0 \\ \sqrt{2m_1 E} \end{pmatrix}$$

□

(b) Do the same for spin down states.
Solution:
Similarly for the spin down states we get

$$u(1) = \frac{m_1}{\sqrt{E-0}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{E}{m_1} \end{pmatrix} \qquad \Rightarrow u^\dagger(1) = \sqrt{m_1} \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}$$

$$u(2) = \frac{m}{\sqrt{E-p}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{E-p}{m} \end{pmatrix} = \begin{pmatrix} 0 \\ m/\sqrt{E-p} \\ 0 \\ \sqrt{E-p} \end{pmatrix} \qquad \lim_{m\to 0} u(2) = \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix}$$

Similarly we get

$$[\bar{u}(1)\gamma^\mu u(2)] = \begin{pmatrix} \sqrt{2m_1 E} \\ 0 \\ 0 \\ \sqrt{2m_1 E} \end{pmatrix}$$

□

(c) Calculate the vector of values for $s_2 = -1/2$

4. **(SMIN 5.12)** For the single-particle Dirac equation Hamiltonian

$$\hat{H} = -i\gamma^i \partial_i + m$$

(a) Compute the commutator of Hamiltonian operator with the z component of the angular momentum operator $\left[\hat{H}, \hat{L}_2\right]$, where

$$\hat{\boldsymbol{L}} \equiv \boldsymbol{r} \times \boldsymbol{p}$$

Solution:
Writing $-i\partial_i = p_i$ we get

$$\hat{H} = -i\gamma^i \partial_i + m = \gamma^i p_i$$

Since m is scalar it commutes with the L operator so we get

$$\left[\hat{H}, \hat{L}_z\right] = \left[\gamma^i p_i, L_z\right] = \left[\gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z, xP_x - yP_x\right]$$

But using the commutation relations $[x_i, p_j] = i\delta_{ij}$ and $[p_i, p_j] = 0$ we get

$$\begin{aligned}
&\left[\gamma^1 p_x, xP_y - yP_x\right] = \left[\gamma^1 p_x, xp_y\right] - \left[\gamma^1 p_x, yp_x\right] = \gamma^1(-ip_y) = -i\gamma^1 p_y \\
&\left[\gamma^2 p_y, xP_y - yP_x\right] = \left[\gamma^2 p_y, xp_y\right] - \left[\gamma^2 p_y, yp_x\right] = \gamma^2(ip_x) = i\gamma^2 p_x \\
&\left[\gamma^3 p_z, xP_y - yP_x\right] = 0
\end{aligned}$$

Thus the commutation becomes

$$\left[\hat{H}, \hat{L}_z\right] = -i\gamma^1 p_y + i\gamma^2 P_x$$

Which is the required commutation relation of Hamiltonian and the z component of L. $\square$

(b) now consider the spin operator

$$\hat{\boldsymbol{S}} = \frac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

Compute the z component of $\hat{S}^z u_-(p)$
Solution:
The operator $\hat{S}^z$ and the state $u_-(p)$ are

$$\hat{S}^z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad u_-(p) = \frac{m}{\sqrt{E-p}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{E-p}{m} \end{pmatrix}$$

Apply ying the operator simply scales by $\frac{1}{2}$ and flips the sign of secon and last component yielding

$$\hat{S}^z u_-(p) = -\frac{1}{2} u_-(p)$$

This is the required state after operation. $\square$

(c) Compute $\left[\hat{H}, \hat{S}_2\right]$.
Solution:
Writing Hamiltonian as

$$\hat{H} = -i\gamma^i \partial_i + m = \gamma^i p_i$$

Since the operator p_i commute with the 4×4 matrices S and γ

$$\left[\gamma^i p_i, S_z\right] = \left[\gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z, S_z\right] = p_x\left[\gamma^1, S_z\right] + p_y\left[\gamma^2, S_z\right] + p_z\left[\gamma^3, S_z\right]$$

Using the commutation relations $\left[S_z, \gamma^1\right] = i\gamma^2$ and $\left[S_z, \gamma^2\right] = -i\gamma^1$ we obtain

$$\left[\hat{H}, \hat{S}_z\right] = -i\gamma^2 p_x + i\gamma^1 p_y$$

Which is the required commutation relation. $\square$

(d) Coomparint your answers, derive a conserved quantity for the free fermions.
Solution:
Clearly from two parts above $\left[\hat{H}, \mathbf{L} + \mathbf{S}\right] = 0$ thus the conserved operator is $\mathbf{L} + \mathbf{S}$. For free fermion of state $\psi(p)$ the conserved quantity is

$$(\mathbf{L} + \mathbf{S})\psi(p)$$

The eigenvalue of this operator gives the conserved quantity. $\square$