# PHYS T580: The Standard Model

Homework #4

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1. **(SMIN 4.1)** Consider a rectangle.

   (a) List all the possible unique transformations that can be performed that will leave it looking the same as it did initially.
   **Solution:**
   The possible transformations that leave the rectangle looking the same are
      i. Leaving where it is (I).
      ii. Rotation through 180° (R).
      iii. Flipping along the vertical axis through mid points of $A$ & $B$ and $C$ & $D$ ($F_y$).
      iv. Flipping along the horizontal axis through mid points of $A$ & $C$ and $B$ & $D$ ($F_x$).

   □

   (b) Construct the multiplication table for your set of transformations.
   **Solution:**
   The multiplication table for the transformations is

| $\circ$ | I | R | $F_x$ | $F_y$ |
|---|---|---|---|---|
| $I$ | I | R | $F_x$ | $F_y$ |
| $R$ | R | I | $F_y$ | $F_x$ |
| $F_x$ | $F_x$ | $F_y$ | I | $R$ |
| $F_y$ | $F_y$ | $F_x$ | $R$ | I |

   □

   (c) Does this set have the properties of a group?
   **Solution:**
   From the multiplication table it is clear that the element satisfy closure. The element $I$ acts as the identity. Each elements are the inverses of themselves. And associativity is evidently followed. This proves that the elements form a group. □

2. **(SMIN 4.2)** Quaternions are a set of objects that are an extension of imaginary numbers except that there are three of them $i$, $j$ and $k$, with the relations

$$i^2 = j^2 = k^2 = ijk = -1$$

.

   (a) Construct the smallest group possible that contains all the quarternions.
   **Solution:**
   Closure of the group requires that at least, $i, j, k$ and $-1$ to be the members of the group. Since $i^2 = i \circ i = -1$, $i$ can't be the identity of the group. Similarly $j$ and $k$ can't be identity of the group. That leaves $-1$ as the only candidate for the identity of the group. If we can satisfy other requirement of group, then $i, j, k$ and $-1$ will form a group with $-1$ as the identity.

If we define $-1 \circ -1 = -1$, which doesn't violate any of the given requirements, $-1$, works as the identity element.

Since $i^2 = i \circ i = -1$ and $-1$ is identity, $i$ by definition becomes the inverse of itself. Similarly $j$ and $k$ are inverses of themselves. So the group is

$$G\left(\{-1, i, j, k\}, \circ\right)$$

□

(b) Compute the commutation relation $[j, i]$.
**Solution:**
The commutator of a group is defined as

$$[j, i] = j^{-1} i^{-1} j i$$

Where $i^{-1}$ and $j^{-1}$ are the inverses of $i$ and $j$ respectively. Also since $ijk = -1$. Multiplying by $i^{-1}$ on the left gives $jk = i$ and multiplying by $k^{-1}$ on the right gives $ij = k$. From (2a) we have $i^{-1} = i$ and $j^{-1} = j$

$$[j, i] = j^{-1} i^{-1} j i = jiji = j(ij)i = j(k)i = (jk)i = ii = -1$$

Since the commutator is identity element of the group, this group is abelian so that the elements commute. □

(c) Construct a multiplication table for the quarternions.
**Solution:**
The multiplication table becomes

| $\circ$ | $-1$ | $i$ | $j$ | $k$ |
|---|---|---|---|---|
| $-1$ | $-1$ | $i$ | $j$ | $k$ |
| $i$ | $i$ | $-1$ | $k$ | $j$ |
| $j$ | $j$ | $k$ | $-1$ | $i$ |
| $k$ | $k$ | $j$ | $i$ | $-1$ |

This is the required multiplication table. □

3. **(SMIN 4.6)** Expand the series $e^{-i\theta\sigma_2}$ explicitly and reduce to common trigonometric, algebraic or hypergeometric functions.
**Solution:**
The SU(2) rotation matrix with generator $\sigma_2$ is $\boldsymbol{M}(\theta) = e^{-i\theta\sigma_2}$. Expanding it out as a Taylor series gives

$$e^{-i\theta\sigma_2} = 1 - i\theta\sigma_2 - \sigma_2^2\frac{\theta^2}{2!} + i\sigma_2^3\frac{\theta^3}{3!} + \sigma_2^4\frac{\theta^4}{4!} - \dots$$

Since for the Pauli matrices $\sigma_i^2 = 1$ which implies that for odd powers the Pauli matrices are the matrices themselves and for even power they reduce to identity, thus we can write

$$\begin{aligned}
e^{-i\theta\sigma_2} &= 1 - i\theta\sigma_2 - \frac{\theta^2}{2!} + i\sigma_2\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \dots \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots - i\theta\sigma_2 + i\sigma_2\frac{\theta^3}{3!} - \dots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) - i\sigma_2\left(\theta - \frac{\theta^3}{3!} + \dots\right) \\
&= \cos\theta - i\sigma_2\sin\theta
\end{aligned}$$

Writing out the explicit matrix form for identity $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ and $\sigma_2 = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$ we get

$$\begin{aligned}\boldsymbol{M} = e^{-i\theta\sigma_2} &= \cos\theta\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - i\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\sin\theta\\ &= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\end{aligned}$$

This is the required $2\times 2$ matrix representation of $e^{-i\theta\sigma_2}$. □

4. **(SMIN 4.10)** Consider a universe consisting of a complex field defined by two components

$$\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

The Lagrangian takes the form

$$\mathcal{L} = \partial^\mu\Phi^\dagger\partial_\mu\Phi - m^2\Phi^\dagger\Phi.$$

In some sense, there are four fields at work here, $\phi_2$, $\phi_2^*$, $\phi_1$ and $\phi_1^*$. But for the purpose of this problem, you should generally think $\Phi$ and $\Phi^\dagger$ as representing the two different fields. Since each is a $2-D$ vector, there are still four degrees of freedom.

(a) Consider a rotation in SU(2) in $\theta^1$ direction ($\sigma_x$). Expand $\boldsymbol{M}$ as infinite series, and express as a $2\times 2$ matrix of only trigonometric functions of $\theta^1$.
**Solution:**
The SU(2) rotation matrix with generator $\sigma_x$ is $\boldsymbol{M}(\theta) = e^{-i\theta\sigma_x}$. Expanding it out as a Taylor series gives

$$e^{-i\theta\sigma_x} = 1 - i\theta\sigma_x - \sigma_x^2\frac{\theta^2}{2!} + i\sigma_x^3\frac{\theta^3}{3!} + \sigma_x^4\frac{\theta^4}{4!} - \dots$$

Since for the Pauli matrices $\sigma_i^2 = 1$ which implies that for odd powers the Pauli matrices are the matrices themselves and for even power they reduce to identity, thus we can write

$$\begin{aligned}e^{-i\theta\sigma_x} &= 1 - i\theta\sigma_x - \frac{\theta^2}{2!} + i\sigma_x\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \dots\\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots - i\theta\sigma_x + i\sigma_x\frac{\theta^3}{3!} - \dots\\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) - i\sigma_x\left(\theta - \frac{\theta^3}{3!} + \dots\right)\\ &= \cos\theta - i\sigma_x\sin\theta\end{aligned}$$

Writing out the explicit matrix form for identity $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ and $\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ we get

$$\begin{aligned}\boldsymbol{M} = e^{-i\theta\sigma_x} &= \cos\theta\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - i\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\sin\theta\\ &= \begin{pmatrix}\cos\theta & -i\sin\theta\\ -i\sin\theta & \cos\theta\end{pmatrix}\end{aligned}$$

This is the required $2\times 2$ matrix representation of $SU(2)$ representing rotation in $\theta^1$ direction. □

(b) Verify numerically that your matrix ($i$) is unitary and ($ii$) has a determinant of 1.
**Solution:**

Checking for Unitarity

$$
\begin{aligned}
\boldsymbol{M}\boldsymbol{M}^\dagger &= \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta + (-i\sin\theta)(i\sin\theta) & (i\cos\theta\sin\theta) + (-i\cos\theta\sin\theta) \\ (-i\cos\theta\sin\theta) + (i\cos\theta\sin\theta) & \cos^2\theta + (-i\sin\theta)(i\sin\theta) \end{pmatrix} \\
&= \begin{pmatrix} \sin^2\theta + \cos^2\theta & 0 \\ 0 & sin^2\theta + \cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}
\end{aligned}
$$

This shows the matrix is unitary. Checking for determinant

$$
\det\{\boldsymbol{M}\} = \begin{vmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{vmatrix} = \cos\theta\cos\theta - (-i\sin\theta)(-i\sin\theta) = \cos^2\theta + \sin^2\theta = 1
$$

The determinant of the matrix is also 1. □

(c) Compute a general expression for the current associated with the rotations in $\theta^1$.
**Solution:**
This Lagrangian is clearly invariant under the transformation $\Phi \to \boldsymbol{M}\Phi$. The generator of which is $\sigma_2$ thus the conserved current is

$$
\begin{aligned}
\mathcal{L} &= g^{\mu\nu}\partial_\nu\Phi^\dagger\partial_\mu\Phi - m^2\Phi^\dagger\Phi & \qquad \mathcal{L} &= \partial^\mu\Phi^\dagger\partial_\mu\Phi - m^2\Phi^\dagger\Phi \\
\Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi^\dagger)} &= g^{\mu\nu}\partial_\mu\Phi = \partial^\nu\Phi & \qquad \Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} &= \partial^\mu\Phi^\dagger
\end{aligned}
$$

$$
\begin{aligned}
J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\frac{\mathrm{d}\Phi}{\mathrm{d}\epsilon} + \frac{\mathrm{d}\Phi^\dagger}{\mathrm{d}\epsilon}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^\dagger)} \\
&= \partial^\mu\Phi^\dagger(-i\sigma_2\Phi) + (i\sigma_2\Phi^\dagger)\partial^\mu\Phi \\
&= i\sigma_2\left(-\left(\partial^\mu\Phi^\dagger\right)\Phi + \Phi^\dagger\partial^\mu\Phi\right)
\end{aligned}
$$

This gives the expression for conserved current. □