PHYS T580: The Standard Model

Homework #1

Prakash Gautam

April 9, 2018

1. **(SMIN 3.3)** A particle of mass m and charge q in an electromagnetic field has a Lagrangian

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - q(\phi - \dot{\boldsymbol{r}} \cdot \boldsymbol{A}),$$

where ϕ is the scalar potential, and $\boldsymbol{A}$ is the vector potential.

(a) Suppose (just for the moment) that the potential fields are not explicit functions of x. Use Noether's theorem to compute the conserved quantity of the electromagnetic Lagrangian.
Solution:
Writing the Lagrangian in cartesian coordinate system we get

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\left(\phi(y, z) - \dot{x}A_x - \dot{y}A_y - \dot{z}A_z\right)$$

Since the lagrangian is invariant under translation in $x \to x + \epsilon$ the conserved quantity is

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\frac{\mathrm{d}x}{\mathrm{d}\epsilon} = \qquad m\dot{x} + qA_x$$

So the conserved quantity if the potential fields are independent of x is $m\dot{x} + qA_x$. □

(b) More generally assume that the potential fields vary in space and time. What are the Euler-Lagrange equations for this Lagrangian corresponding to particle position x^i?
Solution:
If the potential fields depend upon space and time the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^i\dot{x}^i\right) - q\left(\phi(x^i) - \dot{x}^iA_i\right)$$

The Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) = \frac{\partial \mathcal{L}}{\partial x^i}$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}^i + qA_i\right) = -q\left(\frac{\partial \phi}{\partial x^i} - \dot{x}^j\frac{\partial A_j}{\partial x^i}\right)$$

$$\Rightarrow \qquad m\ddot{x}^i + q\frac{\mathrm{d}A_i}{\mathrm{d}t} = -q\left(\frac{\partial \phi}{\partial x^i} - \dot{x}^j\frac{\partial A_j}{\partial x^i}\right)$$

These are the required Euler-Lagrange equations. □

(c) Solve the previous solution explicitly for $m\ddot{x}$. Express your final answer as a combination of $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.
Solution:

Specifically for $x^i = x$ the above expression becomes

$$\begin{aligned}
m\ddot{x} + q\frac{\mathrm{d}A_x}{\mathrm{d}t} &= -q\left(\frac{\partial \phi}{\partial x} - \dot{x}\frac{\partial A_x}{\partial x} - \dot{y}\frac{\partial A_y}{\partial x} - \dot{z}\frac{\partial A_z}{\partial x}\right) \\
m\ddot{x} + q\left(\frac{\partial A_x}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial A_x}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial A_x}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right) &= -q\left(-E_x - \dot{x}\frac{\partial A_x}{\partial x} - \dot{y}\frac{\partial A_y}{\partial x} - \dot{z}\frac{\partial A_z}{\partial x}\right) \\
m\ddot{x} + q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right) &= qE_x + q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}\right) \\
m\ddot{x} &= qE_x + q\left(\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right) \\
m\ddot{x} &= qE_x + q\left(\dot{y}B_z - \dot{z}B_y\right) \\
m\ddot{x} &= qE_x + q\left(\boldsymbol{r} \times \boldsymbol{B}\right)_x
\end{aligned}$$

This is the required equation of motion for the x coordinate under given Lagrangian. □

2. **(SMIN 3.6)** We've seen that a real valued scalar field may be expanded as a plane-wave solution:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right]$$

Computer the total anisotropic stress $\int d^3x T^{ij}$ where $i \neq j$, for a real-valued field by integrating over the stress-energy tensor.

Solution:

The stress-energy tensor is

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial^\nu \phi - g^{\mu\nu}\mathcal{L}$$

For $i \neq j$ $g^{ij} = 0$ so the Tensor reduces to

$$T^{ij} = \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)}\partial^j \phi = \partial_i \phi \partial^j \phi$$

For a complex scalar field given we have

$$\partial_i \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}\partial_i\left(e^{-ip\cdot x}\right) + c^*_{\boldsymbol{p}}\left(\partial_i e^{ip\cdot x}\right)\right] = i\int \frac{d^3p}{(2\pi)^3}\frac{p^i}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right]$$

And similarly the

$$\partial^j \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}\partial^j\left(e^{-ip\cdot x}\right) + c^*_{\boldsymbol{p}}\left(\partial^j e^{ip\cdot x}\right)\right] = -i\int \frac{d^3p}{(2\pi)^3}\frac{p_j}{\sqrt{E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right]$$

Thus the product is

$$\begin{aligned}
T^{ij} &= i\int \frac{d^3p}{(2\pi)^3}\frac{p^i}{\sqrt{2E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right] \cdot -i\int \frac{d^3p}{(2\pi)^3}\frac{p_j}{\sqrt{2E_p}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{p^i}{\sqrt{2E_p}}\frac{q_j}{\sqrt{2E_q}}\left[c_{\boldsymbol{p}}e^{-ip\cdot x} + c^*_{\boldsymbol{p}}e^{ip\cdot x}\right]\left[c_{\boldsymbol{q}}e^{-iq\cdot x} + c^*_{\boldsymbol{q}}e^{iq\cdot x}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{p^i q_j}{\sqrt{4E_pE_q}}\left[c_{\boldsymbol{p}}c_{\boldsymbol{q}}e^{-i(p+q)\cdot x} + c_{\boldsymbol{p}}c^*_{\boldsymbol{q}}e^{-i(p-q)\cdot x} + c^*_{\boldsymbol{p}}c_{\boldsymbol{q}}e^{i(p-q)\cdot x} + c^*_{\boldsymbol{p}}c^*_{\boldsymbol{q}}e^{i(p+q)\cdot x}\right]
\end{aligned}$$

Integrating this quantity over the volume yields

$$\int T^{ij}d^3x = \int d^3x\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{p^i q_j}{\sqrt{4E_pE_q}}\left[c_{\boldsymbol{p}}c_{\boldsymbol{q}}e^{-i(p+q)\cdot x} + c_{\boldsymbol{p}}c^*_{\boldsymbol{q}}e^{-i(p-q)\cdot x} + c^*_{\boldsymbol{p}}c_{\boldsymbol{q}}e^{i(p-q)\cdot x} + c^*_{\boldsymbol{p}}c^*_{\boldsymbol{q}}e^{i(p+q)\cdot x}\right] \tag{1}$$

Since the integration operator is commutative for independent variables the volume integral reduces the complex integral to Dirac delta functions

$$\int e^{i(p-q)\cdot x} d^3x = (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$$

So if we perform q integral any the integral is nonzero only when the q value is equal to p as

$$\int \frac{d^3q}{(2\pi)^3} \frac{q_j}{\sqrt{2E_q}} c_{\boldsymbol{p}} c_{\boldsymbol{q}} \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})(2\pi)^3 = \frac{p_j}{\sqrt{2E_p}} c_{\boldsymbol{p}} c_{\boldsymbol{p}}$$

So (1) reduces to

$$\begin{aligned}\int T^{ij} d^3x &= \int \frac{d^3p}{(2\pi)^3} \frac{p^i p_j}{\sqrt{4E_p^2}} \left[c_{\boldsymbol{p}} c_{-\boldsymbol{p}} + c_{\boldsymbol{p}} c^*_{\boldsymbol{p}} + c^*_{\boldsymbol{p}} c_{\boldsymbol{p}} + c^*_{\boldsymbol{p}} c^*_{\boldsymbol{p}}\right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{p^i p_j}{2E_p} \left[c_{\boldsymbol{p}} c_{-\boldsymbol{p}} + c_{\boldsymbol{p}} c^*_{\boldsymbol{p}} + c^*_{\boldsymbol{p}} c_{\boldsymbol{p}} + c^*_{\boldsymbol{p}} c^*_{\boldsymbol{p}}\right]\end{aligned}$$

This is the required anisotropic stress required. □

3. **(SMIN 3.8)** We might suppose, that a vector field has Lorentz-invariant Lagrangian

$$\mathcal{L} = \partial_\mu A^\nu \partial_\nu A^\mu - m^2 A_\mu A^\mu$$

(a) Compute the Euler-Lagrange equations of for this Lagrangian.
Solution:
The Euler-Lagrange equation are

$$\begin{aligned}\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) &= \frac{\partial \mathcal{L}}{\partial A^\nu} \\ \partial_\mu \partial_\nu A^\mu &= -m^2 A_\mu \\ \partial^\mu \partial_\nu A^\mu &= -m^2 A^\mu\end{aligned}$$

These are the required Euler-Lagrange equations. □

(b) Assume a plane-wave solution for the vector field

$$A^\mu = \int \frac{d^3p}{(2\pi)^3} \epsilon^\mu \frac{1}{\sqrt{2E_p}} \left[a_p e^{-ip\cdot x} + a^*_p e^{ip\cdot x}\right]$$

where we haven't specified polarization state(s) e^μ explicitly.
Develop an explicit relationship between polarization, the momentum of the field, and the mass. What condition does this impose for a massless vector particle?
Solution:
The stress-energy tensor is

$$\begin{aligned}T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\alpha)} \partial^\nu A^\alpha - g^{\mu\nu} \mathcal{L} \\ &= \partial_\alpha A^\mu \partial^\nu A^\alpha - g^{\mu\nu} \left(\partial_\mu A^\alpha \partial_\nu A^\alpha - m^2 A_\alpha A^\alpha\right)\end{aligned}$$

Using the given A^μ vector with every in this expression we get

$$\int \frac{d^3p}{(2\pi)^3} \epsilon^\mu \epsilon^\alpha \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{4E_p E_q}} \left[a_p e^{-ip\cdot x} + a^*_p e^{ip\cdot x}\right] \left[a_q e^{-iq\cdot x} + a^*_q e^{iq\cdot x}\right]$$

Using smiilar development in (1) we get

$$\int T^{\mu\nu} d^3x = \int \frac{d^3p}{(2\pi)^3} \epsilon^\mu \epsilon^\nu \frac{1}{\sqrt{4E_pE_p}} \left[(p^\alpha p_\alpha + p^\alpha p_\alpha - m^2)a_p a_p^*\right]$$
$$= \int \frac{d^3p}{(2\pi)^3} \epsilon^\mu \epsilon^\nu \frac{2E_p^2 - p^2 - m^2}{2E_p} a_p a_p^*$$

□

(c) What is the energy density of the vector field?
Solution:
The stress-energy tensor is in the form For energy density $\mu = 0, \nu = 0$

$$T^{00} = \int \frac{d^3p}{(2\pi)^3} \epsilon^0 \epsilon^0 \frac{2E_p^2 - 2p^2 - m^2}{2E_p} a_p a_p^*$$

This gives the required energy density. □

4. **(SMIN 3.9)** We will often describe multiplets of scalar fields,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

where ϕ_1 and ϕ_2 is each, in this case, a real-valued scalar field for example

$$\mathcal{L}\frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi - \frac{1}{2}m^2 \Phi^T \Phi$$

is a compact way of describing two free scalar fields with identical masses. This Lagrangian is symmetric under the transformation

$$\Phi \to (I - i\theta X)\,\Phi$$

where X is some unknown 2×2 matrix, and θ is assumed to be small.

(a) What is the transformation Φ^T? Show that $\Phi^T\Phi$ remains invariant under this transformation.
Solution:
Taking the transpose of Φ we get

$$\Phi^T \to \Phi^T \left(I - i\theta X^T\right)$$

The quantity $\Phi^T\Phi$ after transformation is

$$\Phi^T\Phi \to \Phi^T \left(I - i\theta X^T\right) \cdot (I - i\theta X)\,\Phi$$
$$= \left(\Phi^T - i\theta \Phi^T X^T\right)(\Phi - i\theta X \Phi)$$
$$= \Phi^T\Phi - i\theta\Phi^T X\Phi - i\theta\Phi^T X^T\Phi - \theta^2\Phi^T XX\Phi$$
$$= \Phi^T\Phi - i\theta\Phi^T \left(X + X^T\right)\Phi - \mathcal{O}(\theta^2)$$

But this transformation preserves the product $\Phi^T\Phi$ only if $X^T = -X$ so that the middle term vanishes.

$$\Phi^T \to \Phi^T(1 + i\theta X)$$

In either of these case

$$\Phi^T\Phi \to \Phi^T\Phi - \mathcal{O}(\theta^2) \approx \Phi^T\Phi$$

This shows that this transformation preserves $\Phi^T\Phi$. □

(b) What is the conserved current in this system?
Solution:
Writing out the Lagrangian in terms of ϕ_1 and ϕ_2 we get

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\begin{pmatrix}\partial_\mu\phi_1 & \partial_\mu\phi_2\end{pmatrix}\begin{pmatrix}\partial^\mu\phi_1 \\ \partial_\mu\phi_2\end{pmatrix} - \frac{1}{2}m^2\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \\ &= \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right) - \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right)\end{aligned}$$

For this transformation the transformed scalar field elements of the matrix are

$$\begin{pmatrix}\phi_1' \\ \phi_2'\end{pmatrix} = \begin{pmatrix}1 - i\theta X_{00} & -i\theta X_{01} \\ -i\theta X_{10} & 1 - i\theta X_{11}\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = \begin{pmatrix}\left(1 - i\theta X_{00}\right)\phi_1 & -i\theta X_{01}\phi_2 \\ -i\theta X_{10}\phi_1 & \left(1 - i\theta X_{11}\right)\phi_2\end{pmatrix}$$

Thus the derivative of ϕ_1' and ϕ_2' with θ become

$$\frac{\mathrm{d}\phi_1'}{\mathrm{d}\theta} = -iX_{00}\phi_1 - iX_{01}\phi_2 \qquad \frac{\mathrm{d}\phi_2'}{\mathrm{d}\theta} = -iX_{10}\phi_1 - iX_{11}\phi_2$$

The conserved current now becomes

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\frac{\mathrm{d}\phi_1'}{\mathrm{d}\theta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\frac{\mathrm{d}\phi_2'}{\mathrm{d}\theta} \\ &= \frac{1}{2}\left(\partial^\mu\phi_1\left(-iX_{00}\phi_1 - iX_{01}\phi_2\right) + \partial^\mu\phi_2\left(-iX_{01}\phi_1 - iX_{11}\phi_2\right)\right)\end{aligned} \tag{2}$$

This gives explicit expression for conserved current in terms of matrix elements of unknown matrix X. □

(c) As we will see, for the particular case described int his problem, the elements of X are

$$X = \begin{pmatrix}0 & i \\ -i & 0\end{pmatrix}.$$

Compute the conserved current in terms of ϕ_1 and ϕ_2 explicitly.
Solution:
If this matrix is taken then $X_{00} = 0, X_{11} = 0, X_{01} = i$ and $X_{10} = -i$. Substuting these in (2) we get,

$$J^\mu = \frac{1}{2}\left[(\partial^\mu\phi_1)\phi_2 - (\partial^\mu\phi_2)\phi_1\right]$$

This gives the explicit expression of conserved current for this particular transformation matrix. □

5. **(SMIN 3.11)** Find the derivative of the following sentence.

$$f(x) = \frac{\sin x}{\cos x}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\frac{\mathrm{d}\phi_1'}{\mathrm{d}\theta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\frac{\mathrm{d}\phi_2'}{\mathrm{d}\theta}$$