PHYS T580: The Standard Model

Homework #2

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1. **(SMIN 2.3)** Consider two particles of equal mass m connected by a spring of constant k and confined tomove in one dimension. The entire ssytem moves without friction. At equilibrium the spring has length L.

   (a) Write down the Lagrangian of this system as a function of x_1 and x_2 and their derivatives. Assume $x_2 > x_1$.
   Solution:
   The kinetic energy of each mass is $\frac{1}{2}m\dot{x_1}^2$ and for the second mass is $\frac{1}{2}m\dot{x_2}^2$. The total compression in the spring is $x_2 - x_1 - L$ so the total potential energy in the spring is $V = \frac{1}{2}k\left(x_2 - x_1 - L\right)^2$. So the lagrangian of the system becomes

   $$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k\left(x_2 - x_1 - L\right)^2$$

   This is the required Lagrngian. □

   (b) Write the Euler-Lagrange equation for this system.
   Solution:

   $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_1}\right) = \frac{\partial\mathcal{L}}{\partial x_1} \quad \Rightarrow m\ddot{x}_1 = k\left(x_2 - x_1 - L\right)$$
   $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_2}\right) = \frac{\partial\mathcal{L}}{\partial x_2} \quad \Rightarrow m\ddot{x}_2 = -k\left(x_2 - x_1 - L\right)$$

   These are the required Euler-Lagrange equation of the system. □

   (c) Make the change of variables

   $$\Delta \equiv x_2 - x_1 - L \qquad X = \frac{1}{2}\left(x_2 + x_2\right)$$

   Write the Lagrangian in these new variables.
   Solution:
   Eliminating x_1 and x_2 between the two transformation variables Δ and X we get

   $$x_2 - x_1 = \Delta + L \qquad x_1 + x_1 = 2X$$
   $$2x_2 = 2X + \Delta + L \qquad \Rightarrow x_2 = X + \frac{1}{2}\left(\Delta + L\right) \qquad \dot{x_2} = \dot{X} + \frac{1}{2}\dot{\Delta}$$
   $$2x_1 = 2X - \Delta - L \qquad \Rightarrow x_2 = X - \frac{1}{2}\left(\Delta + L\right) \qquad \dot{x_1} = \dot{X} - \frac{1}{2}\dot{\Delta}$$

Using these variables the lagrangian becomes

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}m\left(\dot{X}+\frac{1}{2}\dot{\Delta}\right)^2 + \frac{1}{2}m\left(\dot{X}-\frac{1}{2}\dot{\Delta}\right)^2 + \frac{1}{2}k\Delta^2 \\
&= \frac{1}{2}m\left(2\dot{X}^2+\dot{\Delta}^2\right) + \frac{1}{2}k\Delta^2
\end{aligned}
$$

This is the lagrangian in the transformed coordinate system. □

2. **(SMIN 2.7)** Show that the complex Lagrangian $\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^*$ is algebraically identical to

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_2^2\phi_2^2
$$

if $m_1 = m_2 1 = m$ and

$$
\phi = \left(\frac{\phi_1+\phi_2}{2}\right) + i\left(\frac{\phi_1-\phi_2}{2}\right)
$$

Solution:
Assuming the scalar fields ϕ_1 and ϕ_2 are real valued function. The complex field and its conjugate are

$$
\begin{aligned}
\phi &= \left(\frac{\phi_1+\phi_2}{2}\right) + i\left(\frac{\phi_1-\phi_2}{2}\right) \qquad \phi^* = \left(\frac{\phi_1+\phi_2}{2}\right) - i\left(\frac{\phi_1-\phi_2}{2}\right) \\
\Rightarrow \phi\phi^* &= \frac{1}{4}\left[(\phi_1+\phi_2)^2 + (\phi_1-\phi_2)^2\right] = \frac{1}{4}\left(\phi_1^2+\phi_2^2+2\phi_1\phi_2+\phi_1^2+\phi_2^2-2\phi_1\phi_2\right) \\
&= \frac{1}{2}(\phi_1^2+\phi_2^2)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu\phi\partial^\mu\phi^* &= \partial_\mu\left(\frac{\phi_1+\phi_2}{2} + i\frac{\phi_1+\phi_2}{2}\right)\partial^\mu\left(\frac{\phi_1+\phi_2}{2} - i\frac{\phi_1+\phi_2}{2}\right) \\
&= \frac{1}{2}\left(\partial_\mu\phi_1+\partial_\mu\phi_2+i\partial_\mu\phi_1+i\partial_\mu\phi_2\right)\frac{1}{2}\left(\partial^\mu\phi_1+\partial^\mu\phi_2-i\partial^\mu\phi_1-i\partial^\mu\phi_2\right) \\
&= \frac{1}{4}(\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_1\partial^\mu\phi_2-i\partial_\mu\phi_1\partial^\mu\phi_1-i\partial_\mu\phi_1\partial^\mu\phi_2 \\
&\quad+\partial_\mu\phi_2\partial_\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2-i\partial_\mu\phi_2\partial^\mu\phi_1-i\partial_\mu\phi_2\partial^\mu\phi_2 \\
&\quad+i\partial_\mu\phi_1\partial^\mu\phi_1+i\partial_\mu\phi_1\partial^\mu\phi_2+\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_1\partial^\mu\phi_2 \\
&\quad+i\partial_\mu\phi_2\partial^\mu\phi_1+i\partial_\mu\phi_2\partial^\mu\phi_2+\partial_\mu\phi_2\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2) \\
&= \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2\right)
\end{aligned}
$$

Substuting these back in to the complex Lagrngian we get

$$
\begin{aligned}
\mathcal{L} &= \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^* \\
&= \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2\right) - m^2\frac{1}{2}(\phi_1^2+\phi_2^2) \\
&= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_2^2\phi_2^2
\end{aligned}
$$

This shows the two Lagrangian are equivalent. □

3. **(SMIN 2.9)** Consider a lagrangian of real-valued scalar field:

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{6}c_3\phi^3.
$$

(a) Is this Lagrangian Lorentz invariant? It is invariant under C, P, and T transformations individually?
Solution:
Since every term in the lagrangian is a scalar it is trivially Lorentz invariant. As ϕ is real valued scalar its complex conjugate is itself $\phi^* = \phi$ since the $\hat{\mathbf{C}}$ transformation transforms ϕ to ϕ^* which are identical so the Lagrngian is invariant under $\hat{\mathbf{C}}$ transformation.
It is not invariant under $\hat{\mathbf{P}}$ and $\hat{\mathbf{T}}$ transformation. □

(b) What is the dinemsion of c_3?
Solution:
Since the dimension of Lagrangian density is $[E]^4$ and the dimensionality fo ϕ is $[E]$ the dimensionality of c_3 is $[E]^3$ □

(c) What is Euler-Lagrange equation for field?
Solution:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$
$$\partial_\mu \left(\partial^\mu \phi \right) = -m^2 \phi - \frac{c_3}{2} \phi^2$$

This is the required Euler-Lagrange equation for the given Lagrangian density. □

(d) Ignoring the c_3 contribution, a free-field solution may be written

$$\phi_0(x) = Ae^{-ip\cdot x} + A^* e^{ip\cdot x}$$

for a complex coefficient A. Consider a lowest-order contribution for $\phi_1 \ll A$ to a peturbation such that $\phi(x) = \phi_0 + \phi_1$. Derive a dynamical eqution for ϕ_1.
Solution:

$$\partial_\mu \partial^\mu \left(\phi(x) \right) + m^2 \phi(x) + \frac{c_3}{2} \phi^2 = 0$$

Substuting $\phi = \phi_0 + \phi_1$

$$\Rightarrow \qquad \partial_\mu \partial^\mu \left(\phi_0(x) + \phi_1(x) \right) + m^2 \left(\phi_0(x) + \phi_1(x) \right) + \frac{1}{2} c_3 (\phi_0 + \phi_1)^2 = 0$$
$$\Rightarrow \qquad \partial_\mu \partial^\mu \phi_1(x) + m^2 \phi_1(x) + \frac{1}{2} c_3 \phi_0^2 \left(1 + \frac{\phi_1}{\phi_0} \right)^2 = -\partial_\mu \partial^\mu \phi_0 - m^2 \phi_0$$
$$\Rightarrow \qquad \partial_\mu \partial^\mu \phi_1(x) + m^2 \phi_1(x) + \frac{1}{2} c_3 \phi_0^2 \left(1 + 2\frac{\phi_1}{\phi_0} \right) = -\partial_\mu \partial^\mu \phi_0 - m^2 \phi_0$$

The first term in RHS of above expression is

$$\begin{aligned}
\partial_\mu \partial^\mu \phi_0 &= g^{\mu\nu} \partial_\nu (\partial_\mu \phi_0) \\
&= g^{\mu\nu} \partial_\nu \left((-ip_\mu) Ae^{-ip\cdot x} + (ip_\mu) A^* e^{ip\cdot x} \right) \\
&= \partial_\nu \left((-ip^\nu) Ae^{-ip\cdot x} + (ip^\nu) A^* e^{ip\cdot x} \right) \qquad (\text{ Distributing } g^{\mu\nu}) \\
&= (-p^\nu p_\nu) Ae^{-ip\cdot x} + (-p^\nu p_\nu) A^* e^{ip\cdot x} \\
&= -(E^2 - |\vec{p}|^2) \phi_0
\end{aligned}$$

□