PHYS T580: The Standard Model

Homework #1

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1. **(SMIN 1.1)** Consider a vector field in three-dimensional cartesian space:

$$u^i = \begin{pmatrix} xy \\ x^2+2 \\ 3 \end{pmatrix}$$

(a) Compute the components of $\partial_j u^i$.
(b) Compute $\partial_i u^i$
(c) Compute $\partial_j \partial^j u^i$.

Solution:
Since for three-dimensional cartesian space the indices run from 1 through 3,

$$\partial_j u^i = \begin{pmatrix} \partial_1 u^1 & \partial_1 u^2 & \partial_1 u^3 \\ \partial_2 u^1 & \partial_2 u^2 & \partial_2 u^3 \\ \partial_3 u^1 & \partial_3 u^2 & \partial_3 u^3 \end{pmatrix} = \begin{pmatrix} y & 2x & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\partial_i u^i = \partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3 = y + 0 + 0 = y$$

Since for cartesian space the metric elements are

$$\partial_i = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}, \qquad g_{ij} = g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \partial^j = g^{ij}\partial_i = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\Rightarrow \partial_j \partial^j = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \nabla^2 \qquad \Rightarrow \qquad \partial_j \partial^j u^i = \nabla^2 \begin{pmatrix} xy \\ x^2+2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Each of the components are thus calculated. □

2. **(SMIN 1.4)** Express the following quantities in naural units, in the form (# GeV)^n.

(a) The current energy density of universe : $\sim 1 \times 10^{-26} kg/m^3$.
(b) 1 angstrom
(c) 1 nanosecond
(d) 1 gigaparsec $\simeq 3 \times 10^{25} m$
(e) The luminosity of the sun $\simeq 4 \times 10^{26} W$

Solution:

$$\frac{1kg}{m^3} = \left(\frac{c^2 J}{\left(\frac{1}{c}s\right)^3} \right) = \left(\frac{c^2 J}{\left(\frac{1}{c}\frac{1}{\hbar}J^{-1}\right)^3} \right) = c^5 \hbar^3 J^4 = c^5 \hbar^3 \left(\frac{1}{e \times 10^9} GeV \right)^4 = \left(\frac{c^{5/4}\hbar^{3/4}}{e \times 10^9} GeV \right)^4$$

$$\frac{1 \times 10^{-26} kg}{m^3} = \left(\frac{1 \times 10^{-26/4} c^{5/4} \hbar^{3/4}}{e \times 10^9} GeV \right)^4 = (1.01 \times 10^{-11} GeV)^4$$

$$1m = \frac{1}{c}s = \frac{1}{c\hbar}J^{-1} = (c\hbar J)^{-1} = \left(\frac{c\hbar}{e \times 10^9}GeV\right)^{-1}$$

$$1\,\text{Å} = 1 \times 10^{-10}m = \left(\frac{1 \times 10^{10}c\hbar}{e \times 10^9}GeV\right)^{-1} = (1.24 \times 10^{-5}GeV)^{-1}$$

$$1Gpc \simeq 3 \times 10^{25}m = \left(\frac{1 \times 10^{-25}c\hbar}{3 \cdot e \times 10^9}GeV\right)^{-1} = (4.13 \times 10^{-41}GeV)^{-1}$$

$$1s = \frac{1}{\hbar}J^{-1} = \left(\frac{\hbar}{e \times 10^9}GeV\right)^{-1} \quad \Rightarrow \quad 1ns = 1 \times 10^{-9} = \left(\frac{1 \times 10^9 \hbar}{e \times 10^9}GeV\right)^{-1} = (4.41 \times 10^{-15}GeV)^{-1}$$

$$1W = \frac{1J}{1s} = \frac{1J}{\frac{1}{\hbar}J^{-1}} = \hbar J^2 = \left(\frac{\hbar^{1/2}}{e \times 10^9}GeV\right)^2$$

$$L_\odot \simeq 4 \times 10^{26}W = \left(\frac{(4 \times 10^{26})^{1/2}\hbar^{1/2}}{e \times 10^9}GeV\right)^2 = (3.21 \times 10^6 GeV)^2$$

□

3. **(SMIN 1.6)** Consider a 4-vector

$$A^\mu = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

(a) Compute $A \cdot A = A^\mu A_\mu$.
Solution:
For minkowski space the metric is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \Rightarrow A_\mu = g_{\mu\nu}A^\nu = \begin{pmatrix} 2 & -3 & 0 & 0 & 0 \end{pmatrix}$$

$$A \cdot A = A^\mu A_\mu = 2 \cdot 2 + (3) \cdot (-3) = -5$$

Thus the dot product is -5. □

(b) What are the components of $A^{\bar{\mu}}$ if you rotate the coordinate frame around the z-axis through an angle $\theta = \pi/3$?
Solution:
The transformation matrix for rotation around z-axis at an angle $\theta = \frac{\pi}{3}$ is

$$\Lambda^{\bar{\mu}}_{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi/3 & \sin \pi/3 & 0 \\ 0 & -\sin \pi/3 & \cos \pi/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\nu}A^\nu = \begin{pmatrix} 2 \\ 3/2 \\ -3\sqrt{3}/2 \\ 0 \end{pmatrix}$$

These are the required transformed components. □

(c) For your answer in part (3b), verify that $A^{\bar{\mu}}A_{\bar{\mu}}$ is the same as in part (3a).
Solution:

$$g^{\bar\mu\bar\nu} = \Lambda^{\bar\mu}_{\mu}\Lambda^{\bar\nu}_{\nu} g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad g_{\bar\mu\bar\nu} = g^{\bar\mu\bar\nu} \qquad A_{\bar\mu} = g_{\bar\mu\bar\nu}A^{\bar\nu} = \begin{pmatrix} 2 \\ -3/2 \\ 3\sqrt{3}/2 \\ 0 \end{pmatrix}$$

$$A_{\bar\mu}A^{\bar\mu} = 2\cdot 2 + {}^3\!/_2 \cdot (-{}^3\!/_2) + \left(-3^{\sqrt{3}}\!/_2\right) + 3^{\sqrt{3}}\!/_2 = 4 - \frac{9}{4} - \frac{27}{4} = -5$$

The inner product is -5 as required. □

(d) What are the components $A^{\bar\mu}$ if you boost the frame (from (3a)) a speed $v = 0.6$ in x-direction?
Solution:
For $v = 0.6 = 3/5$ the "gamma factor" is $\gamma = 1/\sqrt{1-.6^2} = 1.25 = 5/4$ and thus $v\gamma = 0.75 = 3/4$ The transformation matrix under this boost is

$$\Lambda^{\bar\mu}_{\mu} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5/4 & 3/4 & 0 & 0 \\ 3/4 & 5/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^{\bar\mu} = \Lambda^{\bar\mu}_{\nu}A^{\nu} = \begin{pmatrix} 19/4 \\ 21/4 \\ 0 \\ 0 \end{pmatrix}$$

which are the required components under boost. □

(e) For your answer in part (3d), verify that $A^{\bar\mu}A_{\bar\mu}$ is the same as in part (3a).
Solution:
Since the Minkowski metric is invariant under boost, the transformed metric is $g_{\bar\mu\bar\nu} = g_{\mu\nu}$

$$g_{\bar\mu\bar\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_{\bar\mu} = g_{\bar\mu\bar\nu}A^{\bar\nu} = \begin{pmatrix} 19/4 \\ -21/4 \\ 0 \\ 0 \end{pmatrix}, \quad A^{\bar\mu}A_{\bar\mu} = {}^{19}\!/_4 \cdot {}^{19}\!/_4 + {}^{21}\!/_4 \cdot (-{}^{21}\!/_4) = -5$$

The inner product is -5 as in (3a). □

4. **(SMIN 1.10)** Consider a scalar field

$$\phi(x) = 2t^2 - 3x^2$$

(a) Compute the components of $\partial_\mu\phi$.
(b) Compute the components of $\partial^\mu\phi$.
(c) Compute $\partial_\mu\partial^\mu\phi$. (This operation is the **d'Alembertian operator**).
Solution:

$$\partial_\mu\phi(x) = \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\phi(x) = \begin{pmatrix} 2t \\ -6x \\ 0 \\ 0 \end{pmatrix} \qquad \partial^\mu = g^{\mu\nu}\partial_\nu = \begin{pmatrix} \frac{\partial}{\partial t} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix}, \quad \Rightarrow \partial^\mu\phi(x) = \begin{pmatrix} 2t \\ 6x \\ 0 \\ 0 \end{pmatrix}$$

The operator

$$\partial_\mu\partial^\mu = \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial t} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix} = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) \equiv \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\partial_\mu\partial^\mu\phi(x) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi(x) = 4 + 6 = 10$$

Thus the d'Alembertian operator on the given scalar function $\phi(x)$ is 10. □

5. **(SMIN 1.13)** An excited hydrogen atom emits a $10.2eV$ Lyman$-\alpha$ photon.
   (a) What is the momentum of the photon? (Express in natural units.)
   (b) As Newton's third law remains in force, what is the kinetic energy of the recoiling ground state hydrogen atom?
   (c) What is the recoil speed of proton.

Solution:
For a photon $E = p$ so the momentum of the photon is $10.2eV$.

If Newton's law remain ins force, then the recoiling ground state atom has the same momentum as the outgoing photon. So the recoiling momentum of ground state atom is $10.2eV$.

Mass of proton is $9.38 \times 10^8 eV$, since the mass of electron is negligible coompared to proton let us assume the proton carries all the momentum so,

$$p = \gamma m v \quad \Rightarrow \quad \frac{v}{\sqrt{1-v^2}} = \frac{p}{m} \quad \Rightarrow \quad v = \frac{1}{\sqrt{1+\left(\frac{m}{p}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{9.38\times 10^8}{10.2}\right)^2}} = 1.08 \times 10^{-8} \equiv 3.26 m/s$$

So the recoil speed of proton is $1.08 \times 10^{-8} \equiv 3.26 m/s$. □