# PHYS 631: General Relativity

Homework #6

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1. **(Schutz 11.7)** A clock is in a circular orbit at $r = 10M$ in a Schwarzschild metric.

   (a) How much time elapses on the clock during one orbit?
   **Solution:**
   The proper time and the interval are related by the expression $d\tau^2 = ds^2$. For circular orbit $dr = d\phi = 0$ so we get

$$d\tau^2 = ds^2 = g^{\phi\phi}(U_\phi)^2 d\phi^2 \implies d\tau = U^\phi d\phi$$

   But for circular orbit the quantity $p_\phi = m\tilde{L}$ thus we obtain

$$U^\phi = \frac{1}{m}p^\phi = g^{\phi\phi}\frac{p_\phi}{m} = \frac{1}{r^2}\tilde{L}$$

   The quantity $\tilde{L}^2 = \frac{Mr}{1-3M/r}$ substituting these we get

$$U^\phi = \frac{1}{r^2}\sqrt{\frac{Mr}{1-\frac{3M}{r}}}$$

   The time elapsed is given by

$$\tau = \int_0^\tau d\tau = \int_0^{2\pi} \frac{1}{U^\phi} d\phi = \int_0^{2\pi} \sqrt{\frac{r^4(1-3M/r)}{Mr}} d\phi$$

   Noting that, the integrand is independent of $\phi$, for circular orbit at $r = 10M$ we obtain

$$\tau = 2\pi\sqrt{\frac{1000M^3}{M}\left(1 - \frac{3M}{10M}\right)} = 2\pi 10\sqrt{7}M$$

   This is the time elapsed in the clock. □

   (b) It sends out a signal to ta distant observer once each orbit. What time interval does the distant observer measure between receiving any two signals?
   **Solution:**
   The time elapsed for a distant observer is the coordinate time for the Schwarzschild metric. If it sends signal every orbit, the time elapsed for distant observer is the coordinate time for one full orbit. To find the coordinate time we have to get expression for $dt = f(\vec{x})d\phi$, where $t$ is the coordinate time. From the definition of the $\phi$ component of four velocity

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = U^\phi = \frac{p^\phi}{m} = g^{\phi\phi}\frac{p_\phi}{m} = g^{\phi\phi}\tilde{L} = \frac{1}{r^2}\tilde{L}$$

   Similarly from the $0th$ component of four velocity we get

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = U^0 = \frac{p^0}{m} = g^{00}\frac{p_0}{m} = g^{00}(-\tilde{E}) = \frac{\tilde{E}}{1-2M/r} \tag{1}$$

Combining these two we get

$$\frac{\mathrm{d}t}{\mathrm{d}\phi} = \frac{dt/d\tau}{d\phi/d\tau} = \left(\frac{r^3}{M}\right)^{1/2}$$

Now that we have obtained the functional form connecting the coordinate time and azimuthal angle. We can integrate to find

$$t = 2\pi\left(\frac{r^3}{M}\right)^{\frac{1}{2}}$$

For $r = 10M$ we obtain

$$t = 2\pi\sqrt{\frac{r^3}{M}} = 2\pi\sqrt{\frac{1000M^3}{M}} = 2\pi 10\sqrt{10}M. \tag{2}$$

This is the coordinate time that passes for one orbit which is the time measured by the distant observer and is also the time it elapses for distant observer for a complete revolution. □

(c) A second clock is located at rest at $r = 10$ next to the orbit of the first clock. How much time elapses on it between successive passes of the orbiting clock?
**Solution:**
The time is dilated in the orbiting clock by the time dilation factor which is simply

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{-1/g_{00}} = \sqrt{1 - 2M/r}$$

Now the proper time is given by

$$\tau = \sqrt{1 - \frac{2M}{r}}t$$

Substituting the coordinate time expression form (**??**) we get

$$\tau = \sqrt{1 - \frac{2M}{r}}2\pi\sqrt{\frac{r^3}{M}} \tag{3}$$

Substituting $r = 10M$ we obtain

$$\tau = 2\pi\sqrt{8}M.$$

This gives the time elapsed in the stationary clock as the clock makes one orbit. □

(d) Calculate (**??**) again in seconds for an orbit at $r = 6M$ where $M = 14M_\odot$. This is the minimum fluctuation time we expect in the X-ray spectrum of Cyg X-1: why?
**Solution:**
For r = 6M substituting $r = 6M$ in (**??**) we get

$$t = 2\pi\sqrt{216}M = 12\pi\sqrt{6}\cdot 14M_\odot$$

The mass of sun $M_\odot = 1.9\times 10^{30}kg = 1.476\times 10^3 m$. Substation these

$$t = \frac{12\pi\sqrt{6}\cdot 1.476\times 10^3}{3\times 10^8} = 0.00636s = 6.36\times 10^{-3}s$$

this is the time elapsed. □

(e) If the orbiting 'clock' is the twin Artemis, in the obit in (**??**), how much does she age during the time her twin Diana lives 40years far from the black hole and at rest with respect to it?
**Solution:**
We already have for a circular orbit from (**??**) we have

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\tilde{E}}{1 - \frac{2M}{r}}$$

For a stable orbit in the Schwarzschild metric we have

$$\tilde{E} = \frac{1 - 2M/r}{\sqrt{1 - 3M/r}}$$

Substituting we get

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - 3M/r}}$$

Solving this differential equation we get

$$\int d\tau = \int \sqrt{1 - \frac{3M}{r}} dt$$

Setting $r = 6M$ gives

$$\tau = t\sqrt{\frac{1}{2}}$$

For $t = 40yr$ we get

$$\tau = \frac{40}{\sqrt{2}} = 28.28yr$$

This is the age of Artemis when her twin Diana lives 40yr. □

2. **(Schutz 11.21)** A particle of $m \neq 0$ falls radially toward the horizon of a Schwarzschild black hole of mass $M$. The geodesic it follows has $\tilde{E} = 0.95$

(a) Find the proper time required to reach $r = 2M$ from $r = 3M$.
**Solution:**
We have for a massive object the radial motion near the Schwarzschild metric satisfies:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right)$$

The proper time is then given by

$$\tau = \int \frac{dr}{\sqrt{\tilde{E}^2 - 1 + \frac{2M}{r}}} \tag{4}$$

Making substituting $\alpha = \tilde{E}^2 - 1$ we get the following integral

$$\tau = \int \frac{dr}{\sqrt{\alpha + \frac{2M}{r}}}$$

The integral is

$$\tau = \left[\frac{2M\sqrt{r}}{\alpha\sqrt{2M + \alpha r}} - \frac{2M \operatorname{asinh}\left(\frac{\sqrt{2}\sqrt{\alpha}\sqrt{r}}{2\sqrt{M}}\right)}{\alpha^{\frac{3}{2}}} + \frac{r^{\frac{3}{2}}}{\sqrt{2M + \alpha r}}\right]_{3M}^{2M}$$

$$= \frac{3\sqrt{3}M^{\frac{3}{2}}}{\sqrt{3M\alpha + 2M}} - \frac{2\sqrt{2}M^{\frac{3}{2}}}{\sqrt{2M\alpha + 2M}} + \frac{2\sqrt{3}M^{\frac{3}{2}}}{\alpha\sqrt{3M\alpha + 2M}} - \frac{2\sqrt{2}M^{\frac{3}{2}}}{\alpha\sqrt{2M\alpha + 2M}} + \frac{2M \operatorname{asinh}(\sqrt{\alpha})}{\alpha^{\frac{3}{2}}} - \frac{2M \operatorname{asinh}\left(\frac{\sqrt{6}\sqrt{\alpha}}{2}\right)}{\alpha^{\frac{3}{2}}}$$

Substituting $\alpha = 0.95^2 - 1$ we obtain

$$\tau = 1.1917M$$

This is the required time for the journey from $3M$ to $2M$ for a infilling particle. $\square$

(b) Find the proper time required to reach $r = 0$ from $r = 2M$.
**Solution:**
Similar to previous part the proper time required is

$$\begin{aligned}\tau &= \left[\frac{2M\sqrt{r}}{\alpha\sqrt{2M+\alpha r}} - \frac{2M\,\mathrm{asinh}\left(\frac{\sqrt{2}\sqrt{\alpha}\sqrt{r}}{2\sqrt{M}}\right)}{\alpha^{\frac{3}{2}}} + \frac{r^{\frac{3}{2}}}{\sqrt{2M+\alpha r}}\right]_{2M}^{0} \\ &= \frac{2\sqrt{2}M^{\frac{3}{2}}}{\sqrt{2M\alpha+2M}} + \frac{2\sqrt{2}M^{\frac{3}{2}}}{\alpha\sqrt{2M\alpha+2M}} - \frac{2M\,\mathrm{asinh}\left(\sqrt{\alpha}\right)}{\alpha^{\frac{3}{2}}}\end{aligned}$$

Substituting $\alpha = 0.95^2 - 1$ we obtain

$$\tau = 1.3745M$$

This is the required time for the journey from $2M$ to center for a infalling particle. $\square$

(c) Find, on the Schwarzschild coordinate basis, its four-velocity components ate $r = 2.001M$.
**Solution:**
For radially moving object $U^\phi = U^\theta = 0$. The timelike component is given by

$$U^0 = -g^{00}\tilde{E} = \frac{\tilde{E}}{1-\frac{2M}{r}} = \frac{0.95}{1-\frac{2}{2.001}} = 1900.95$$

The radial component can be obtained by reusing (**??**) as

$$(U^r)^2 = \tilde{E}^2 - \left(1-\frac{2M}{r}\right) \qquad \Longrightarrow \qquad U^r = \sqrt{0.95^2 - 1 + \frac{2}{2.001}} = 0.949$$

Thus the four velocity is

$$U^\mu = \begin{pmatrix}1900.95\\0.949\\0\\0\end{pmatrix}$$

This is the component of four velocity at $r = 2.001M$ $\square$

(d) As it passes $2.001M$ , it sends a photon out radially to a distant stationary observer. Compute the redshift of the photon when it reaches the observer.
**Solution:**
The energy observed by the distant observer is given by

$$E_{\text{obs}} = -U\cdot p = -\left(-U_0 p^0 + U_r p^r\right)$$

We can calculate the component $p^r$ by using the fact that photon is massless. Since for photon we have

$$p^2 = -m^2 = 0$$

Expanding the dot product of the momentum we get

$$g_{tt}(p^t)^2 + g_{rr}(p^r)^2 = 0$$

But we have $p_t = E$ so we can rewrite this as

$$g^{tt}(p_t)^2 + g_{rr}(p^r)^2 = 0 \qquad \implies p^r = \sqrt{-\frac{g^{tt}}{g_{rr}}}p_t = \sqrt{\frac{1-\frac{2M}{r}}{1-\frac{2M}{r}}}E = E$$

Substituting this in the observed energy expression we get

$$E_{\text{obs}} = (U_0 p^0 - U_r p^r) = \left(U^0 p_0 - U_r p^r\right) = E(U^0 - U_r)$$

But $U_r = g_{rr}U^r = \frac{U^r}{1-2M/r}$ substituting $U^0 = 1900.95$ and $U^r = 0.949$ we get

$$E_{\text{obs}} = E\left(1900.95 + \frac{0.949}{1-\frac{2}{2.001}}\right) = 3801.95E$$

This gives the observed energy of the photon. So the redshift factor is simply

$$z = \frac{E_{\text{obs}} - E}{E} = \frac{3801.95E - E}{E} = 3800.95$$

Which is the required redshift factor. □

3. Using the relations that we derived in class:

$$a_{\text{y-stretching}} = \frac{2M}{r^3}\Delta y \qquad \text{and} \qquad a_{\text{x-compressing}} = \frac{M}{r^3}\Delta x$$

Throughout this problem, assume that you dropped from rest at infinity.

(a) Find the smallest black hole in which you could survive long enough to pass the event horizon.
**Solution:**
In the event horizon $r = 2M$. The maximum acceleration that human can survive is $a_{max} \sim 9g$. So we get

$$a_{\max} = \frac{M}{(2M)^3}\Delta x \implies M = \sqrt{\frac{\Delta x}{4a_{\max}}}$$

Substituting $\Delta x \sim 1m$ $g \sim 10\frac{m}{s^2}$ we get

$$M = \sqrt{\frac{1}{360}} = \frac{1}{6\sqrt{10}}s$$

Since $1s = 299792458m$ and $1m = 1.34 \times 10^{27}kg$ we get

$$M = \frac{1}{6\sqrt{10}} \cdot 299792458 \cdot 1.34 \times 10^{27} = 2.12 \times 10^{34}kg = 1.07 \times 10^4 M_\odot$$

This is the most massive black hole one can survive near the event horizon. □

(b) For a $1M_\odot$ black hole, how long does it take between the time you feel mildly uncomfortable (tidal force between head and feet is $2g$) and you die? This should be in proper time, of course.
**Solution:**
The tidal force will stretch so we have from the given stretching expression

$$a_{\text{y-stretching}} = \frac{2M}{r^3}\Delta y \implies r = \left(\frac{2M\Delta y}{a}\right)^{\frac{1}{3}}$$

For just being 'uncomfortable' $a = 2g$ gives

$$r = \left(\frac{2M_\odot \Delta y}{20}\right)^{\frac{1}{3}}$$

Substituting $M_\odot = 1.98 \times 10^{30} kg$ and $\Delta y \sim 0.5m$

$$r = 4.56 \times 10^9 (s^2 kg)^{1/3}$$

Substituting $1s = 299792458m$ and $1kg = 7.42 \times 10^{-28}m$ we get

$$r = 4.56 \times 10^9 (299792458^2 \cdot 7.42 \times 10^{-28})^{1/3} = 1.85 \times 10^6 m$$

For dying $a = 9g$ we get through similar process

$$r = 2.76 \times 10^9 (s^2 kg)^{1/3} = 2.76 \times 10^9 (299792458^2 \cdot 7.42 \times 10^{-28})^{1/3} = 1.12 \times 10^6 m$$

The proper time to travel between these two distance can be obtained by the expression as in Equation. (**??**) above

$$\tau = \int \frac{dr}{\sqrt{\tilde{E}^2 - 1 + \frac{2M}{r}}}$$

Here $\tilde{E}$ is proportional to initial energy for simplicity assuming $\tilde{E} = 1$ we get

$$\tau = \int \frac{1}{\sqrt{2M}} \sqrt{r} dr = \frac{1}{\sqrt{2M}} \frac{2}{3} r^{\frac{3}{2}} = \frac{\sqrt{2}}{3} \sqrt{\frac{r^3}{M}}$$

Proper time between these two distances is

$$\tau = \left[ \frac{\sqrt{2}}{3} \sqrt{\frac{r^3}{M}} \right]_{r_1}^{r_2}$$

For $M = 1M_\odot$ we get

$$\tau = \left[ \frac{\sqrt{2}}{3} \sqrt{\frac{r^3}{1.98 \times 10^{30}}} \right]_{1.85\times 10^6}^{1.12\times 10^6} = 4.45 \times 10^{-7} \left( \frac{m^3}{kg} \right)^{1/2}$$

Substituting $1kg = 7.42 \times 10^{-28}m$ and $1m = \frac{1}{299792458}s$ we get

$$\tau = 4.45 \times 10^{-7} \left(1.34 \times 10^{27} m^2\right)^{\frac{1}{2}} = 4.45 \times 10^{-7} \cdot 1.22 \times 10^5 s = 5.44 \times 10^{-2} s$$

This gives the time for mild uncomfortably and death. □

(c) How about a $10M_\odot$

**Solution:**

Repeating the same process for $M = 10M_\odot$ we get

$$r_1 = 9.83 \times 10^9 \left(s^2 kg\right)^{1/3} = 3.98 \times 10^6 m$$
$$r_2 = 5.95 \times 10^9 \left(s^2 kg\right)^{1/3} = 2.41 \times 10^6 m$$
$$\tau = 4.44 \times 10^{-7} \left( \frac{m^3}{kg} \right)^{1/3} = 5.426 \times 10^{-2} s$$

So for a $10M_\odot$ the time interval for the falling person from mild uncomfortability to death is $5.42 \times 10^{-2} s$. □

4. **(Schuts 12.9)**

(a) Show that a photon which propagates in a radial null geodesic of the metric, , has energy $-p_0$ inversely proportional to $R(t)$.
**Solution:**
The given metric is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & R^2(t)r^2 & 0 \\ 0 & 0 & 0 & R^2(t)r^2\sin^2\theta \end{pmatrix}$$

For radial geodesic $U^\phi = U^\theta = 0$. Since photon is massless we get

$$p \cdot p = 0 \implies g^{00}(p_0)^2 + g^{rr}(p_r)^2 = 0$$

Simplifying gives

$$(p_0)^2 = -\frac{g^{rr}}{g^{00}}(p_r)^2 = \frac{R^2(t)}{1-kr^2}(p_r)^2 \tag{5}$$

We now have to find the relationship between $p_r$ and the element of metric. The next relationship comes from the geodesic equation as

$$\dot{p}^\mu = \Gamma^\mu_{\rho\nu}\frac{p^\rho p^\nu}{p^0}$$

Specifically for $\mu = 0$ we get

$$\dot{p}^0 = \Gamma^0_{\rho\nu}\frac{p^\rho p^\nu}{p^0}$$

We need the Christoffel symbols for this. The Christoffel symbols are given by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}\right)$$

The only required Christoffel symbols are $\Gamma^0_{\alpha\beta}$

$$\Gamma^0_{\nu\rho} = \frac{1}{2}g^{00}\left(g_{\nu 0,\rho} + g_{\rho 0,\nu} - g_{\nu\rho,0}\right)$$

Explicitly

$$\Gamma^0_{rr} = \frac{1}{2}g^{00}\left(g_{r0,r} + g_{r0,r} - g_{rr,0}\right) = \frac{1}{2}(-1)\left(-\partial_t\left(\frac{R^2(t)}{1-kr^2}\right)\right) = \frac{\dot{R}(t)R(t)}{1-kr^2}$$

Substituting this in the geodesic equation we get

$$\dot{p}^0 = -\frac{\dot{R}(t)R(t)}{1-kr^2}\frac{p^r p^r}{p^0}$$

But from (**??**) we have $(p^r)^2$ and substituting we get

$$\dot{p}^0 = -\frac{\dot{R}(t)R(t)}{1-kr^2}\frac{(1-kr^2)(p^0)^2}{R^2(t)p^0}$$
$$\dot{p}^0 = -\frac{\dot{R}(t)}{R(t)}p^0$$

This is a differential equation , solving we get

$$\frac{dp^0}{p^0} = -\frac{dR(t)}{R(t)} \qquad \ln(p^0) = -\ln(R(t)) \implies p^0 \propto \frac{1}{R(t)}$$

Lowering the index of $p^0$ in the LHS we get

$$p_0 = g_{00}p^0 = -p^0 \implies p_0 \propto -\frac{1}{R(t)}$$

Which is the required expression. $\square$

(b) Show from this that a photon emitted at time $t_e$ and received a time $t_r$ by observers at rest in the cosmological reference from is redshift ed by

$$1 + z = \frac{R(t_r)}{R(t_e)}$$

**Solution:**
For an observer at rest $v^i = 0 \implies U^i = 0$. Using $U \cdot U = -1$ gives

$$g_{00}(U^0)^2 = -1 \implies U^0 = \sqrt{-\frac{1}{g_{00}}} = 1$$

thus the observed energy is

$$E_{\text{obs}} = -p \cdot U_{\text{obs}} = -p_0 U^0 = -p_0$$

calculating the redshift we get

$$z = \frac{E_{\text{obs}}(t_e) - E_{\text{obs}}(t_r)}{E_{\text{obs}}(t_r)} = \frac{-\frac{1}{R(t_e)} + \frac{1}{R(t_r)}}{-\frac{1}{R(t_r)}} =$$

Simplifying

$$1 + z = 1 + \frac{-\frac{1}{R(t_e)} + \frac{1}{R(t_r)}}{-\frac{1}{R(t_r)}} = \frac{R(t_r)}{R(t_e)}$$

This is the required expression. $\square$

5. **(Schuts 12.20)** Assume that the universe is matter dominated and find the value of $\rho_\Lambda$ that permits the universe to be static.

(a) Because the universe is matter-dominated at the present time, we can take $\rho_m(t) = \rho_0 \left[\frac{R_0}{R(t)}\right]^3$ where the subscript 0 refers to the static solution we are looking for. Differentiate the 'energy' equation

$$\frac{1}{2}\dot{R}^2 = -\frac{1}{2}k + \frac{4}{3}\pi R^2 \left(\rho_m + \rho_\Lambda\right) \tag{6}$$

with respect to time to find the dynamical equation governing a matter dominated universe:

$$\ddot{R} = \frac{8}{3}\pi\rho_\Lambda R - \frac{4}{3}\pi\rho_0 R_0^3 R^{-2}$$

Set this to zero to find the solution

$$\rho_\Lambda = \frac{1}{2}\rho_0$$

For Einstein's static solution, the cosmological constant energy density has to be half of th matter energy density.

**Solution:**
As instructed, differentiating with respect to tie we get

$$\begin{aligned}\frac{1}{2} \cdot 2 \cdot \ddot{R}\dot{R} &= \frac{8}{3}\pi R\dot{R}(\rho_m + \rho_\Lambda) + \frac{4}{3}\pi R^2 \left(\dot{\rho}_m + \dot{\rho}_\Lambda\right) \\ \ddot{R} &= \frac{8}{3}\pi R(\rho_m + \rho_\Lambda) + \frac{4}{3}\pi R^2 \left(\dot{\rho}_m + \dot{\rho}_\Lambda\right)\end{aligned}$$

But the functional form of $\rho_m(t)$ is given differentiating we get

$$\dot{\rho}_m = -3\frac{R_0^3\rho_0\dot{R}}{R^4}$$

And for matter dominated universe $\dot{\rho}_\Lambda = 0$ substituting these

$$\ddot{R}\dot{R} = \frac{8}{3}\pi R\dot{R}\left(\rho_m + \rho_\Lambda\right) - 4\pi\frac{\rho_0 R_0^3\dot{R}}{R^2}$$
$$\ddot{R} = \frac{8}{3}\pi R\left(\rho_m + \rho_\Lambda\right) - 4\pi\frac{\rho_0 R_0^3}{R^2}$$

Which is the required dynamical equation. At current time we have $R = R_0$ so we get

$$\ddot{R} = \frac{8}{3}R_0\left(\rho_0 + \rho_\Lambda\right) - 4\rho_0 R_0 = \frac{8}{3}R_0\rho_\Lambda - \frac{4}{3}R_0\rho_0$$

Setting this equal to zero we get

$$\frac{8}{3}R_0\rho_\Lambda = \frac{4}{3}R_0\rho_0$$

We obtain

$$\rho_\Lambda = \frac{1}{2}\rho_0$$

This is the required expression. □

(b) Put our expression for $\rho_m$ into the right-hand-side of (**??**) to gen an energy-like expression which has a derivative that has to vanish for a static solution. Verify that the above condition of $\rho_\Lambda$ does indeed make the first derivative vanish.

**Solution:**

Substituting $\rho_m$ we obtain

$$\frac{1}{2}\dot{R}^2 = -\frac{1}{2}k + \frac{4}{3}\pi R^2\left(\frac{\rho_0 R_0^3}{R^3} + \frac{1}{2}\rho_0\right)$$

For static solution the second term on the right has to have vanishing derivative because the first being constant has zero derivative already. Checking

$$\frac{\partial}{\partial R}\left[\frac{4}{3}\pi R^2\left(\frac{\rho_0 R_0^3}{R^3} + \frac{1}{2}\rho_0\right)\right] = \frac{4}{3}\pi\rho_0\frac{\partial}{\partial R}\left[\left(\frac{R_0^3}{R} + \frac{R^2}{2}\right)\right] = \frac{4}{3}\pi\rho_0\left[\left(-\frac{R_0^3}{R^2} + R\right)\right]$$

For initial time we have $R = R_0$ this expression evaluates to zero. □

(c) Compute the second derivative of the right-hand-side of (**??**) with respect to $R$ and show that, the static solution, it is positive. This means that the ‘potential‘ is a minimum and *Einstein's static solution is stable.*

**Solution:**

The second derivative is

$$\frac{4}{3}\pi\rho_0\frac{\partial}{\partial R}\left[\left(-\frac{R_0^3}{R^2} + R\right)\right] = \frac{4}{3}\pi\rho_0\left[\left(2\frac{R_0^3}{R^3} + 1\right)\right]$$

For today $R = R_0$ and we get

$$\frac{4}{3}\pi\rho_0\left[\left(2\frac{R_0^3}{R^3} + 1\right)\right] = \frac{4}{3}\pi\rho_0\left[\left(2\frac{R_0^3}{R_0^3} + 1\right)\right] = 4\pi\rho_0$$

For $\rho > 0$ the second derivative is positive. This means the solution is stable. □