PHYS 631: General Realtivity

Homework #5

Prakash Gautam

May 30, 2019

1. Consider a 1+1 dimensional space (t, x) with the metric:

$$g_{\mu\nu} = \begin{pmatrix} -e^{kx} & 0 \\ 0 & 1 \end{pmatrix}$$

where k is a dimensional constant.

(a) This metric has a stress-energy source which is (potentially) non-zero. Knowing nothing else, what is the scaling of the density ρ in terms of k?
Solution:
Since the exponent in the metric has to be dimensionless the dimension of k is

$$[k] = [L] = [M]$$

The dimension of density is

$$[\rho] = \frac{[M]}{[L]^3} = [L]^2$$

From these two expressions

$$\rho \sim k^2$$

So, in terms of dimension only the density has to scale as the square of k. □

(b) Compute all non-zero Christoffel symbols.
Solution:
The non zero derivative of the metric is in terms of x only and the only non zero derivative is

$$g_{tt,x} = -ke^{kx}$$

The Christoffel symbols are given by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma}{}_{,\rho} + g_{\rho\sigma}{}_{,\nu} - g_{\nu\rho}{}_{,\sigma} \right)$$

The non zero Christoffel symbols are

$$\Gamma^x_{tt} = \frac{1}{2} g^{xx} \left(-g_{tt,x} \right) = \frac{1}{2} \cdot (-1) \cdot -ke^{kx} = \frac{1}{2} ke^{kx}$$

The other are

$$\Gamma^t_{tx} = \Gamma^t_{xt} = \frac{1}{2} k$$

These are the required non zero Christoffel symbols. □

(c) A massive particle is instantaneously at rest ate $x = 0$. What is the instantaneous acceleration of the particle?
Solution:
The geodesic equation can be used to calculate the acceleration of the particle. From the geodesic equation we have

$$\frac{\partial U^\mu}{\partial \tau} = -\Gamma^\mu_{\alpha\beta} U^\alpha U^\beta$$

For particle at rest $v^i = 0, \implies U^i = 0$. Using $U \cdot U = -1$ we get

$$(U^0)^2 g_{00} = -1 \qquad U^0 = e^{kx/2}$$

Since the only non zero Christoffel symbols are Γ^t_{tx} and Γ^x_{tt} we get

$$\frac{\partial U^x}{\partial \tau} = -\Gamma^x_{tt} U^0 U^0 = -\frac{1}{2} k e^{kx} e^{kx}$$

At the origin thus $x = 0$ we get

$$\frac{\partial U^x}{\partial \tau} = -\frac{1}{2} k$$

This gives the acceleration of the particle. □

(d) Compute the non-zero components of the Riemann tensor.
Solution:
And the Riemann tensor is given by

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu} - \Gamma^\alpha_{\beta\mu}{}_{,\nu} + \Gamma^\alpha_{\beta\nu}{}_{,\mu}$$

calculating

$$\begin{aligned} R^x{}_{txt} &= \Gamma^x_{\sigma x}\Gamma^\sigma_{tt} - \Gamma^x_{\sigma t}\Gamma^\sigma_{tx} - \Gamma^x_{tx,t} + \Gamma^x_{tt,x} \\ &= -\Gamma^x_{tt}\Gamma^t_{tx} + \Gamma^x_{tt,x} \\ &= -\frac{1}{2}k \cdot \frac{1}{2} k e^{kx} + \frac{1}{2} k^2 e^{kx} \\ &= \frac{1}{4} k^2 e^{kx} \end{aligned}$$

Similarly the other component of Riemann tensor are

$$R^t{}_{xtx} = -\frac{1}{4} k^2$$

The other components are simply the cyclic permutation of the indices. □

(e) What are the non-zero terms in the Ricci Tensor and Ricci Scalar?
Solution:
The components of Ricci tensor in terms of elements of Riemann Tensor are

$$R_{\alpha\beta} = g^{\mu\nu} R^\nu_{\alpha\mu\beta}$$

Specifically for R_{tt} we get

$$\begin{aligned} R_{tt} &= g^{tt} \cancelto{0}{R^t_{ttt}} + g^{xx} R^x_{txt} \\ &= \frac{1}{4} k^2 e^{kx} \end{aligned}$$

Similarly the other component R_{xx} is

$$\begin{aligned}
R_{xx} &= g^{tt}R^t_{xtx} + \cancelto{0}{g^{xx}R^x_{xtx}} \\
&= -e^{-kx}\cdot\frac{1}{4}k^2e^{kx} \\
&= -\frac{1}{4}k^2
\end{aligned}$$

The Ricci scalar can be calculated by contracting the Ricci tensor as

$$R = R^t_t + R^x_x = g^{tt}R_{tt} + g^{xx}R_{xx} = -\frac{1}{4}k^2 - \frac{1}{4}k^2 = -\frac{1}{2}k^2 \tag{1}$$

So the Riccis scalar is $-1/2k^2$. □

(f) What is the Einstein tensor?
Solution:
The components of Einstein tensor are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{2}$$

The first component of this tensor is

$$G_{tt} = R_{tt} - \frac{1}{2}g_{tt}R = \frac{1}{4}k^2e^{kx} + \frac{1}{2}e^{-kx}\cdot -\frac{1}{2}k^2 = 0$$

The other component is

$$G_{xx} = R_{xx} - \frac{1}{2}g_{xx}R = -\frac{1}{4}k^2 - \frac{1}{2}\cdot -\frac{1}{2}k^2 = 0$$

So the Einstein tensor is identically zero. □

2. In the generalized linear metric we found in class:

$$\begin{pmatrix} -1-2\psi & 0 & 0 & 0 \\ 0 & 1-2\phi & 0 & 0 \\ 0 & 0 & 1-2\phi & 0 \\ 0 & 0 & 0 & 1-2\phi \end{pmatrix}$$

where, for a non-relativistically moving source:

$$\nabla^2 = 4\pi(\rho + 3P); \qquad \nabla^2\phi = 4\pi\rho$$

suppose you were in the interior of a spherically symmetric distribution with constant density and fixed equation of sate $w = -\frac{1}{3}$

(a) What is the acceleration on a test particle places a distance r from the center of the cloud. Would it fall inward or outward?
Solution:
Since ψ and ϕ are functions of r only we have non zero derivative of the components of metric only with respect to r. The Christoffel symbols are given by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}\right)$$

The non zero Christoffel symbols are

$$\begin{aligned}
{\Gamma^t}_{tr} = {\Gamma^t}_{rt} &= \frac{1}{2}g^{tt}\left(g_{tt,r} + \cancelto{0}{g_{rt,t}} - \cancelto{0}{g_{tr,r}}\right) \\
&= \frac{1}{2}\frac{-1}{1+2\psi}\left(-2\psi_{,r}\right) \\
&= \frac{\psi_{,r}}{1+2\psi}
\end{aligned}$$

Similarly the other non zero Christoffel symbols are

$$\Gamma^r{}_{tt} = \frac{\psi_{,r}}{1-2\phi} \qquad \Gamma^r{}_{rr} = -\frac{\phi_{,r}}{1-2\phi}$$

For a stationary particle $v^i = 0 \implies U^i = 0$. Using $U \cdot U = -1$ we get

$$(U^0)^2 g_{00} = -1 \implies U^0 = \sqrt{1+2\psi}$$

The geodesic equation can be used to calculate the acceleration of the particle. The geodesic equation is

$$\frac{\partial U^\mu}{\partial \tau} = -\Gamma^\mu_{\alpha\beta} U^\alpha U^\beta$$

The spatial acceleration of the particle is

$$\frac{\partial U^r}{\partial \tau} = \Gamma^r{}_{tt} U^0 U^0 = \frac{\psi_{,r}}{1-2\phi}(1+2\psi)$$

The quantity $\psi_{,r}$ can be calculated by using the fact that the Laplacian of ψ is given. In a spherically symmetric system $\nabla^2 \equiv \frac{\partial^2}{\partial r^2}$, so we get

$$\frac{\partial^2 \psi}{\partial r^2} = 4\pi(\rho + 3P)$$

Integrating once with respect to r we get

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = \psi_{,r} = 4\pi\rho\left(1 + 3\frac{P}{\rho}\right) r$$

Subsisting this in the expression for acceleration we get

$$\frac{\partial U^r}{\partial \tau} = \frac{\psi_{,r}}{1-2\phi}(1+2\psi) = 4\pi\rho(1+3w)r \cdot \frac{1+2\psi}{1-2\phi}$$

Given that $w = -\frac{1}{3}$ we get

$$\frac{\partial U^r}{\partial \tau} = 0 \cdot \frac{1+2\psi}{1-2\phi} = 0$$

So the radial acceleration of the particle is zero. Since for $i \neq r$, $U^i = 0$ and $\Gamma^i{}_{tt} = 0$ all other spatial components of acceleration is zero. So the spatial acceleration of the particle is identically zero. □

(b) What is the acceleration on a photon traveling perpendicular to the cloud also a distance r from the center. Would it be lensed inward or outward?

Solution:

Since the photon is trailing perpendicular to the cloud (in a straight line), we can assume (without loss of generality) the radial and azimuthal components of the velocity are zero, by choosing the direction of travel same as the radial coordinate. So, $U^\theta = 0, U^\phi = 0$ The spatial acceleration of the photon is

$$\frac{\partial U^r}{\partial \tau} = \Gamma^r{}_{tt} U^0 U^0 + \Gamma^r{}_{rr} U^r U^r$$

Subsisting the Christoffel symbols we get

$$\frac{\partial U^r}{\partial \tau} = \frac{\psi_{,r}}{1-2\phi}(U^0)^2 - \frac{\phi_{,}}{1-2\phi}(U^r)^2$$

Again by arguments of previous problem $\psi_{,r} = 0$, so we get

$$\frac{\partial U^r}{\partial \tau} = -\frac{\phi_{,r}}{1-2\phi}(U^r)^2$$

Again, in a spherically symmetric system $\nabla^2 \equiv \frac{\partial^2}{\partial r^2}$, we can similarly obtain $\phi_{,r}$ and ϕ by integrating the Laplacian of ϕ with respect to r once and twice respectively.

$$\phi_{,r} = 4\pi\rho r \qquad \phi = 2\pi\rho r^2$$

Since $\phi \ll 1$, $(U^r)^2 > 0$ and $\phi_{,r} = 4\pi\rho r > 0$ the final expression for the acceleration will turn out to be negative. Thus the acceleration would be inward and hence the photon will be lensed inward. □

3. **(Schutz 8.17)**

   (a) A small planet orbits a static neutron star in a circular orbit whose proper circumference is 6×10^{11} m. The orbital period takes 200days of the planet's proper time. Estimate the mass M of the star.
   Solution:
   For the purpose of estimation we can assume that Newton's laws hold and that the time dilation effect is negligible. In that limit the proper time is just the time measured by observer. From Kepler's third law we have

$$t^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

   If c is the circumference, it is given in terms of radius by, $c = 2\pi r$ subsisting c we get

$$t^2 = \frac{c^3}{2\pi GM} \implies M = \frac{1}{2\pi G}\frac{c^3}{T^2}$$

   So for the given planet

$$t \approx \tau = 200 days = 1.728 \times 10^7 s \qquad c = 6 \times 10^{11} m$$

   So the mass is given by

$$M = \frac{1}{2\pi \cdot 6.672 \times 10^{-11}} \frac{(6 \times 10^{11})^3}{(1.728 \times 10^7)^2} = 1.726 \times 10^{30} kg$$

   So the mass of the neutron star is 1.726×10^{30}kg. □

   (b) Five satellites are placed into a circular orbit around a static black hole. The proper circumferences and proper periods of their orbits are given in a table below. Use the method of 3a to estimate the hole's mass. Explain the results you get for the satellites

circumference	2.5×10^6 m	6.3×10^6 m	6.3×10^7	3.1×10^8 m	6.3×10^9 m
proper period	8.4×10^{-3} s	0.055s	2.1s	23s	2.1×10^3 s

   Solution:
   Using the method of 3a we get

c(m)	t(s)	$\frac{1}{2\pi G}\left(\frac{c^3}{t^2}\right) kg$
2.5×10^6	8.4e-3	5.28×10^{32}
6.3×10^6	0.055	1.97×10^{32}
6.3×10^7	2.1	1.35×10^{32}
3.1×10^8	23	1.34×10^{32}
6.3×10^9	2.1×10^3	1.35×10^{32}

   The obtained value for the mass seem to be converging towards $1.35 \times 10^{32} kg$ with the successive increase in the orbital circumference. So further away the satellite, the Newtonian approximation are more correct. □

4. **(Schutz 8.18)** Consider the field equation with cosmological constant. With Λ arbitrary and $k = 8\pi$.
   (a) Find the Newtonian limit and show that we recover the motion of the planets only if $|\Lambda|$ is very small. Given the radius of Pluto's orbit is 5.9×10^{12} m, set an upper bound on $|\Lambda|$ from solar-system measurements
   Solution:
   The field equation is
$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$
   Newtonian equation of motion is given by
$$\nabla^2 \phi = 4\pi\rho$$
   specifically the first component of field equation
$$G_{00} = 8\pi T_{00} - \Lambda g_{oo}$$
   In the Newtonian limit, since $T_{00} = \rho$ and $g_{00} = -1$, I would expect in field equation term
$$\rho \to \rho + \frac{\Lambda}{8\pi}$$
   I am assuming the limit to Λ comes from the maximum estimation of the mass density ρ in the solar system. Even if the space wre empty and only cosmological constant were present of that value, we would get the orbital radius of Pluto. So maximum value $\Lambda < \rho \times 8\pi$. The measured density of the solar system is in the order
$$\sim 1.3 \times 10^{-22} \frac{g}{cc} = 1.3 \times 10^{-19} \frac{kg}{m^3}$$
   and so maximum Λ should be the same order. □

   (b) By bringing Λ over th the RHS of Schutz eq 8.7 we can regard $-\Lambda g^{\mu\nu}/8\pi$ as the stress-energy tensor of 'empty space'. Given that he observed mass of the region of the universe near our Galaxy would have a density of about 1×10^{-27} kgm^3 if it were uniformly distributed, do you think that a value of $|\Lambda|$ near the limit you established in 4a could have observable consequences for cosmology? Conversely if Λ is comparable to the mass density of the universe, do we need to include it in the equations when we discuss the solar system?
   Solution:
   If Λ is is in the order as predicted in 4a, and the density of galaxy is in the order of 1×10^{-27}kg/m^3 then
$$\Lambda \gg \rho_{\text{galaxy}}$$
   In that case $\rho \to \rho + \frac{\Lambda}{8\pi}$ would be dominated by Λ, so we would have to observable effect of the cosmological constant.
   If the value of Λ is comparable to the density of the universe, then I would still assume that we would need to include in the calculation of solar system. □

5. **(Schuts 10.9)**
   (a) Define a new radial coordinate in terms of the Schwarzschild r by
$$r = \bar{r}\left(1 + \frac{M}{2\bar{r}}\right)^2.$$
   Notice that as $r \to \infty, \bar{r} \to r$, while the event horizon $r = 2M$, where we have $\bar{r} = \frac{1}{2}M$. Show that the metric for spherical symmetry takes the form
$$ds^2 = -\left[\frac{1 - 2M/\bar{r}}{1 + M/\bar{r}}\right]^2 dt^2 + \left[1 + \frac{M}{2\bar{r}}\right]^4 \left[d\bar{r}^2 + \bar{r}^2 d\Omega^2\right]$$

Solution:
The Schwarzschild metric is

$$g_{\mu\nu} = \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & 1/(1-2M/r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The transformation of the elements of metric can be obtained by

$$g_{\bar{\mu}\bar{\nu}} = \Lambda^{\mu}_{\bar{\mu}}\Lambda^{\nu}_{\bar{\nu}}g_{\mu\nu}$$

where the elements of the transformation matrix $\Lambda^{\mu}_{\bar{\mu}}$ are given by

$$\Lambda^{\mu}_{\bar{\mu}} = \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}$$

Since the given metric is diagonal, the only non zero term in the metric $g_{\mu\nu}$ are with $\mu = \nu$. Expanding the metric transformation explicitly as a sum

$$g_{\bar{\mu}\bar{\nu}} = \Lambda^{\mu}_{\bar{\mu}}\Lambda^{\mu}_{\bar{\nu}}g_{\mu\mu}$$

Given the transformation $r \to \bar{r}(1+M/2\bar{r})^2$ and all other coordinates are unchanged we get

$$\begin{aligned}
\Lambda^{r}_{\bar{r}} = \frac{\partial r}{\partial \bar{r}} &= \frac{\partial}{\partial \bar{r}}\left(\bar{r}\left(1+\frac{M}{2\bar{r}}\right)^2\right) \\
&= \left(1+\frac{M}{2\bar{r}}\right)^2 + 2\bar{r}\left(1+\frac{M}{2\bar{r}}\right)\left(-\frac{M}{2\bar{r}^2}\right) \\
&= \left(1+\frac{M}{2\bar{r}}\right)\left(1+\frac{M}{2\bar{r}}-\frac{M}{\bar{r}}\right) \\
&= \left(1+\frac{M}{2\bar{r}}\right)\left(1-\frac{M}{2\bar{r}}\right)
\end{aligned}$$

for all other coordinates $\bar{t} = t, \bar{\phi} = \phi, \bar{\theta} = \theta$ so we get

$$\begin{aligned}
&\Lambda^{\theta}_{\bar{\theta}} = \Lambda^{\phi}_{\bar{\phi}} = \Lambda^{t}_{\bar{t}} = 1 \\
&\Lambda^{\mu}_{\bar{\nu}} = 0 \text{ if } \mu \neq \nu
\end{aligned}$$

Thus expanding the transformation of the metric explicitly we get

$$g_{\bar{t}\bar{t}} = \Lambda^{t}_{\bar{t}}\Lambda^{t}_{\bar{t}}g_{tt} = g_{tt} = -\left(1-\frac{2M}{r}\right)$$

Under the given transformation we have

$$1-\frac{2M}{r} = 1-\frac{2M}{\bar{r}\left(1+M/2\bar{r}\right)^2} = \frac{\bar{r}\left(1-M/2\bar{r}\right)^2-2M}{\bar{r}\left(1+M/2\bar{r}\right)^2} = \frac{\left(1-\frac{M}{2\bar{r}}\right)^2}{\left(1+\frac{M}{2\bar{r}}\right)^2} \tag{3}$$

So under the transformed coordinate system we get

$$g_{\bar{t}\bar{t}} = -\frac{\left(1-M/2\bar{r}\right)^2}{\left(1+M/2\bar{r}\right)^2}$$

The nexe component of the metric is

$$g_{\bar{r}\bar{r}} = \Lambda^{r}_{\bar{r}}\Lambda^{r}_{\bar{r}}g_{rr} = \left[\left(1+\frac{M}{2\bar{r}}\right)\left(1-\frac{M}{2\bar{r}}\right)\right]^2\left(1-\frac{2M}{r}\right)^{-1}$$

Using (3) we get in this expression we get

$$\begin{aligned} g_{\bar{r}\bar{r}} &= \left[\left(1+\frac{M}{2\bar{r}}\right)\left(1-\frac{M}{2\bar{r}}\right)\right]^2 \frac{(1+M/2\bar{r})^2}{(1-M/2\bar{r})^2} \\ &= \left(1+\frac{M}{2\bar{r}}\right)^4 \end{aligned}$$

The next component of the transformed metric is

$$g_{\bar{\theta}\bar{\theta}} = \Lambda^{\theta}_{\bar{\theta}}\Lambda^{\theta}_{\bar{\theta}} g_{\theta\theta} = g_{\theta\theta} = r^2 = \bar{r}^2\left(1+\frac{M}{2\bar{r}}\right)^4$$

The final non zero component is

$$g_{\bar{\phi}\bar{\phi}} = \Lambda^{\phi}_{\bar{\phi}}\Lambda^{\phi}_{\bar{\phi}} g_{\phi\phi} = g_{\phi\phi} = r^2\sin^2\theta = \bar{r}^2\left(1+\frac{M}{2\bar{r}}\right)^4 \sin^2\theta$$

Thus the final transformed metric is

$$g_{\bar{\mu}\bar{\nu}} = \begin{pmatrix} -\frac{(1-M/2\bar{r})^2}{(1+M/2\bar{r})^2} & 0 & 0 & 0 \\ 0 & (1+M/2\bar{r})^4 & 0 & 0 \\ 0 & 0 & \bar{r}^2\left(1+M/2\bar{r}\right)^4 & 0 \\ 0 & 0 & 0 & \bar{r}^2\left(1+M/2\bar{r}\right)^4\sin^2\theta \end{pmatrix}$$

The line element in this metric is given by

$$ds^2 = -\left[\frac{1-2M/\bar{r}}{1+M/\bar{r}}\right]^2 dt^2 + \left[1+\frac{M}{2\bar{r}}\right]^4 \left[d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2\sin^2\theta d\phi^2\right] \tag{4}$$

Which is the required expression. □

(b) Define a quasi-Cartesian coordinates by the usual equations $x = \bar{r}\cos\phi\sin\theta$, $y = \bar{r}\sin\phi\sin\theta$, and $z = \bar{r}\cos\theta$ so that , $d\bar{r}^2 + \bar{r}^2 d\Omega^2 = dx^2 + dy^2 + dz^2$ Thus the metric has been converted into coordinates (x, y, z), which are called isotropic coordinates. Now take the limit as $\bar{r} \to \infty$ and show

$$ds^2 = -\left[1 - \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right] dt^2 + \left[1 + \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right]\left(dx^2 + dy^2 + dz^2\right)$$

Solution:
Under the transformation given

$$d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2\sin^2\theta d\phi^2 = dx^2 + dy^2 + dz^2$$

Under the limit $\bar{r} \to \infty$, the metric element $g_{\bar{t}\bar{t}}$ can be simplified

$$\begin{aligned} g_{\bar{t}\bar{t}} = \frac{(1-M/2\bar{r})^2}{(1+M/2\bar{r})^2} &= -\left(1-\frac{M}{2\bar{r}}\right)^2\left(1+\frac{M}{2\bar{r}}\right)^{-2} \\ &= \left(1-\frac{M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right)\left(1-\frac{M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right) \\ &= \left(1-\frac{M}{\bar{r}} - \frac{M}{\bar{r}} + \frac{M^2}{\bar{r}^2} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right) \\ &= \left(1-\frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right) \end{aligned}$$

Similarly under the approximation g_{rr}

$$g_{\bar{r}\bar{r}} = \left(1 + \frac{M}{2\bar{r}}\right)^4 = 1 + \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)$$

Subsisting this in the line element expression (4) we get

$$ds^2 = -\left[1 - \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right] dt^2 + \left[1 + \frac{2M}{\bar{r}} + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right)\right] \left(dx^2 + dy^2 + dz^2\right)$$

This is the required expression. $\square$

(c) Compute the proper circumference of a circle at radius $\bar{r}$
Solution:
The circumference is given by the total distance traveled by a particle going at a constant distance $\bar{r}$ from the center, which is the length of the line under $\phi : 0 \to 2\pi$ The line element is

$$ds^2 = g_{\bar{\phi}\bar{\phi}} d\phi^2$$

So the total circumference is

$$C = \int_0^{2\pi} \sqrt{\bar{r}^2 \left(1 + \frac{M}{2\bar{r}}\right)^4} d\phi = 2\pi\bar{r} \left(1 + \frac{M}{2\bar{r}}\right)^2$$

Which is the proper circumference. $\square$

(d) Compute the proper distance in traveling from $\bar{r}$ to $\bar{r} + d\bar{r}$.
Solution:
The line element is

$$ds^2 = g_{\bar{r}\bar{r}} d\bar{r}^2$$

The length going from $\bar{r} \to \bar{r} + d\bar{r}$ is

$$ds = \sqrt{g_{\bar{r}\bar{r}}} d\bar{r} = \sqrt{\left(1 + \frac{M}{2\bar{r}}\right)^4} d\bar{r} = \left(1 + \frac{M}{2\bar{r}}\right)^2 d\bar{r}$$

This gives the distance going from $\bar{r} \to \bar{r} + d\bar{r}$. $\square$