PHYS 631: General Relativity

Homework #3

Prakash Gautam

May 1, 2019

1. In a flat space, the metric in spherical coordinates , r, θ, ϕ is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

   (a) Compute all non-zero Christoffel symbols for this system.
   (b) Compute the divergence $V^{\alpha}{}_{;\alpha}$

2. Consider a vectro in 2-d space:

$$\vec{v} = i$$

   starting at $r = 1, \theta = 0$, and moving aroudn the unit circle with constatn $r = 1$, but varying θ. The assumption is that the vector itself should not vary.

   Write, and solve a differential equation describing the changes in the components of $\vec{v}$ as you parallel-transport it around the unit circle.

3. **(Schutz 5.14)** For the tensor whose polar components are $A^{rr} = r^2, A^{r\theta} = r\sin\theta, A^{\theta r} = r\cos\theta, A^{\theta\theta} = \tan\theta$, compute

$$\nabla_\beta A^{\mu\nu} = A^{\mu\nu}{}_{,\beta} + A^{\alpha\nu}\Gamma^{\mu}{}_{\alpha\beta} + A^{\mu\alpha}\Gamma^{\nu}{}_{\alpha\beta}$$

   in polars for all possible indices.

4. **(Schutz 7.3)** Calculate all the Christoffel symbols for the metric,

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)\left(dx^2 + dy^2 + dz^2\right)$$

   , to first order in ϕ. Assume ϕ is a general function of t, x, y and z.

5. A cosmic string is a theoretical construct whcih is infinitely long, and has a mass density per unit length λ. The coordinates describibg the spacetime surrounding a cosmic string are

$$x^\mu = \begin{pmatrix} t \\ R \\ \phi \\ z \end{pmatrix}$$

   and which has a metric:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R^2(1-4\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   (a) Compute the volume element, dV, near the cosmic string.

(b) Compute all non-zero Christoeffl symbols.
(c) Compute the distance between two points separated by $dx^\mu = dR$, and all other coordinates equal to zero. From that, cmptute the distance from the string itself out to t distance $R = 1$
(d) Compute the distance between two points, each $R = 1$ from the string separated by an angle $d\phi$ (with all other $dx^\mu = 0$) Using that, what is the total distance traersed by a particle covering a circular orbit $R = 1$ around the cosmic string?
(e) Compare (5c) and (5d) in the context of the normal relationship between radius and circumference. That is, does $C = 2\pi r$? if not, what should it be replaced with?