PHYS 631: General Relativity

Homework #2

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April 25, 2019

1. A particle in Minkowski space travels along a trajectory:

$$x(\tau) = \alpha\tau^2$$
$$y(\tau) = \tau$$
$$z(\tau) = 0$$

   (a) What are the spacelike components of the 4-velocity, U^i?
   Solution:
   The spacelike components of four velocity is

$$U^i = \frac{\partial x^i}{\partial \tau} = (2\alpha\tau, 1, 0)$$

□

   (b) Using the relation $U \cdot U = -1$, compute U^0.
   Solution:
   The inner product of the four velocity vector $U^\mu = (U^0U^1U^2U^3)$ is

$$\begin{aligned} U \cdot U = -(U^0)^2 + (U^1)^2 + (U^2)^2 + (U^3)^2 = -1 \\ \implies -(U^0)^2 + 4\alpha^2\tau^2 + 1 + 0 = -1 \\ \implies U^0 = \pm\sqrt{2 + (2\alpha\tau)^2} \end{aligned}$$

   This is the timelike component of velocity four vector. □

   (c) What is the 3-velocity of the particle as a function of τ?
   Solution:
   The spacelike components are given by

$$V^i = \frac{U^i}{U^0} = \left(\frac{2\alpha\tau}{\sqrt{2 + (2\alpha\tau)^2}}, \frac{1}{\sqrt{2 + (2\alpha\tau)^2}}, 0 \right)$$

□

2. **(Schutz 3.24)** Give the components of $\binom{2}{0}$ tensor $M^{\alpha\beta}$ as the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}$$

   find:

(a) the components of symmetric tensor $M^{(}\alpha\beta)$ and antisymmetric tensor $M^{[\alpha\beta]}$

Solution:

The symmetric tensor can be written as

$$M^{(\alpha\beta)} = \frac{1}{2}\left(M^{\alpha\beta} + M^{\beta\alpha}\right)$$

When the indices are switched the elements of the tensor are

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

Using this we get the symmetric form

$$M^{(\alpha\beta)} = \begin{bmatrix} 0 & 1 & 1 & 1/2 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1/2 \\ 1/2 & 1 & -1/2 & 0 \end{bmatrix}$$

Similarly the anti symmetric tensor is

$$M^{[\alpha\beta]} = \begin{bmatrix} 0 & 0 & -1 & -1/2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3/2 \\ 1/2 & -1 & -3/2 & 0 \end{bmatrix}$$

These are the required matrices. □

(b) the components of $M^{\alpha}{}_{\beta}$

Solution:

This can be written with the metric tensor as

$$M^{\alpha}{}_{\beta} = g_{\sigma\beta}M^{\alpha\sigma} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}$$

□

(c) the components of M_{α}^{β}

Solution:

This can be written with th metric as

$$M_{\alpha}{}^{\beta} = g_{\alpha\sigma}M^{\sigma\beta} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}$$

□

(d) the components of $M_{\alpha\beta}$

Solution:

The previous tensor can be used to calculate this

$$M_{\alpha\beta} = g_{\sigma\beta}M_{\alpha}{}^{\sigma} = \begin{bmatrix} 0 & -1 & & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}$$

□

3. **(Schutz 3.30)** In some $\mathcal{O}$, the vector U and D have the components

$$U \rightarrow (1 + t^2, t^2, \sqrt{2}t, 0)$$
$$D \rightarrow (x, 5tx, \sqrt{2}t, 0)$$

and the scalar ρ has the value

$$\rho = x^2 + t^2 - y^2$$

(a) Find $U \cdot U$, $U \cdot D$, $D \cdot D$. Is U suitable as four-velocity field? Is D?
Solution:
The components of U_μ are $U_\mu = (-(1 + t^2), t^2, \sqrt{2}t, 0)$ and the components of D_μ are $D_m u = (-x, 5tx, \sqrt{2}t, 0)$ so the dot products are

$$U \cdot U = U^\mu U_m u = (-(1 + t^2)^2 + t^4 + 2t^2 + 0) = -1 - 2t^2 - t^4 + t^4 + 2t^2 = -1$$
$$D \cdot D = D^\mu D_\mu = (-x^2 + 25t^2x^2 + 2t^2 + 0) = x^2(25t^2 - 1) + 2t^2$$
$$U \cdot D = U^\mu D_\mu = -x(1 + t^2) + 5t^3x + 2t^2 = x(5t^3 - t^2 - 1) + 2t^2$$

Since the inner product of U with itself is -1 its is suitable for a four velocity while D is not (except possibly for fixed values of x and t). □

(b) Find the spatial velocity v of a particle whose four-velocity is U,, for arbitrary t. What happens to it in the limits $t \rightarrow 0$ and $t \rightarrow \infty$?
Solution:

$$v^i = \frac{U^i}{U^0} = \left(\frac{t^2}{1 + t^2}, \frac{\sqrt{2}t}{1 + t^2}, 0\right)$$

In the limit $t \rightarrow \infty$ we get $\boldsymbol{v} = (1, 0, 0)$ and in the limit $t \rightarrow 0$ we get $\boldsymbol{v} = (0, 0, 0)$ □

(c) Find U_α for all α
Solution:
With the Minkowski metric the values of U_α is $U_\alpha = (-(1 + t)^2, t^2, \sqrt{2}t, 0)$ □

(d) Find $U^\alpha{}_{,\beta}$ for all α, β
Solution:
The vales are

$$U^\alpha{}_{,\beta} = \frac{\partial U^\alpha}{\partial x^\beta} = \begin{bmatrix} 2t & 0 & 0 & 0 \\ 2t & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

(e) Show that $U^\alpha_{\alpha,\beta} = 0$ for all β. Show that $U^\alpha U_{\alpha,\beta} = 0$ for all β.
Solution:
For various values of β $U_\alpha U^\alpha_{,\beta}$ is

$$\beta = 0 :: U_\alpha U^\alpha_{,0} = \frac{\partial}{\partial t}\left(-(1 + t^2)^2 + t^4 + 2t\right) = -2(1 + t^2) \cdot 2t + 4t^3 + 4t = 0$$
$$\beta = 1 :: U_\alpha U^\alpha_{,1} = \frac{\partial}{\partial x}\left(-(1 + t^2)^2 + t^4 + 2t\right) = 0$$
$$\beta = 2 :: U_\alpha U^\alpha_{,2} = \frac{\partial}{\partial y}\left(-(1 + t^2)^2 + t^4 + 2t\right) = 0$$
$$\beta = 3 :: U_\alpha U^\alpha_{,3} = \frac{\partial}{\partial z}\left(-(1 + t^2)^2 + t^4 + 2t\right) = 0$$

We have $U^\alpha U_\alpha$ is the inner product of $U \cdot U$ and so $U \cdot U = U^\alpha U_\alpha = U_\alpha U^\alpha$ so the expression

$$U^\alpha U_{\alpha,\beta} = (U_\alpha U^\alpha)_{,\beta} = 0, \forall \beta$$

□

(f) Find $D^\beta{}_{,\beta}$
Solution:
It is simply the divergence of vector D so we get

$$D^\beta_{,\beta} = \frac{\partial x}{\partial t} + \frac{\partial 5tx}{\partial x} + \frac{\partial \sqrt{2}t}{\partial y} + \frac{\partial 0}{\partial z} = 5t$$

□

(g) Find $(U^\alpha D^\beta)_{,\beta}$ for all α.
Solution:
The components of tensor $U^\alpha D^\beta$ are

$$U^\alpha D^\beta = \begin{bmatrix} (1+t^2)x & 5tx(1+t^2) & \sqrt{2}t(1+t^2) & 0 \\ t^2x & 5t^3x & \sqrt{2}t^3 & 0 \\ \sqrt{2}tx & 5\sqrt{2}t^2x & 2t^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now the derivatives $(U^\alpha D^\beta)_{,\beta}$ has the components

$$\begin{aligned} &\alpha = 0 : 2tx + 5t(1+t^2) + 0 + 0 = 2tx + 5t(1+t^2) \\ &\alpha - 1 : 2tx + 5t^3 + 0 + 0 = 2tx + 5t^3 \\ &\alpha = 2 : \sqrt{2}x + 5\sqrt{2}t^2 + 0 + 0 = \sqrt{2}x + 5\sqrt{2}t^2 \\ &\alpha = 3 : 0 \end{aligned}$$

So the components are $(U^\alpha D^\beta)_{,\beta} = (2tx + 5t(1+t^2), 2tx + 5t^3, \sqrt{2}x + 5\sqrt{2}t^2)$. □

(h) Find $U_\alpha (U^\alpha D^\beta)_{,\beta}$ and compare result.
Solution:
We have the components of $U_\alpha = (-(1+t^2), t^2, \sqrt{2}t, 0)$ and we have obtained

$$M^\alpha = (U^\alpha D^\beta)_{,\beta} = (2tx + 5t(1+t^2), 2tx + 5t^3, \sqrt{2}x + 5\sqrt{2}t^2)$$

$$\begin{aligned} U_\alpha (U^\alpha D^\beta)_{,\beta} &= U_\alpha M^\alpha \\ &= (-(1+t^2)(2tx + 5t(1+t^2)) + t^2(2tx + 5t^3) + \sqrt{2}t(\sqrt{2}xt + 5\sqrt{2}t^2)) \\ &= -5t \end{aligned}$$

We see that this is equal to $-D^\beta_{,\beta}$ and using the fact that $U_\alpha U^\alpha = -1$ we can rewrite

$$U_\alpha (U^\alpha D^\beta)_{,\beta} = -D^\beta_{,\beta} = (U_\alpha U^\alpha) D^\beta_{,\beta}$$

This shows that the associative property in tensors hold. □

(i) Find $\rho_{,\alpha}$ for all α. Find $\rho^{,\alpha}$ for all α
Solution:
The components are

$$\rho_{,\alpha} = \left(\frac{\partial \rho}{\partial t}, \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z} \right) = (2t, 2x, -2y, 0)$$

The raised version is

$$\rho^{,\beta} = (-2t, 2x, -2y, 0)$$

□

4. **(Schuts 4.17)** We have defined $a^\mu = U^\mu_{,\beta} U^\beta$. Go to the non-relativistic limit and show that

$$a^i = \dot{v}^i + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) v^i$$

Solution:
Writing out the components of the above expression we get

$$a^\mu = \frac{\partial U^\mu}{\partial x^0} U^0 + \frac{\partial U^i}{\partial x^1} U^1 + \frac{\partial U^i}{\partial x^2} U^2 + \frac{\partial U^i}{\partial x^3} U^3$$

The spatial components are

$$a^i = \frac{\partial U^i}{\partial x^0} U^0 + \frac{\partial U^i}{\partial x^1} U^1 + \frac{\partial U^i}{\partial x^2} U^2 + \frac{\partial U^i}{\partial x^3} U^3$$

In the non relativistic limit $U^0 = 1$ and $U^i = v^i$ where v^i is the component of velocity so we obtain

$$a^i = \frac{\partial v^i}{\partial t} + \frac{\partial v^i}{\partial x} v^x + \frac{\partial v^i}{\partial y} v^y + \frac{\partial v^i}{\partial z} v^z$$

This expression can be rearranged into

$$a^i = \dot{v}^i + (v^x \hat{\boldsymbol{i}} + v^y \hat{\boldsymbol{j}} + v^z \hat{\boldsymbol{k}}) \cdot \left(\hat{\boldsymbol{i}} \frac{\partial}{\partial x} + \hat{\boldsymbol{j}} \frac{\partial}{\partial y} + \hat{\boldsymbol{k}} \frac{\partial}{\partial z} \right) v^i$$

Since the nabla operator is the middle term in above expression we get

$$a^i = \dot{v}^i + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) v^i$$

This is the required expression. □

5. Consider a stationary, ideal fluid of the form:

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

For the moment, you should assume that the stress-energy tensor is constant in time and throughout space

(a) Compute the stress energy tensor $T^{\bar{\mu}\bar{\nu}}$ in a frame moving at a speed, v with respect to th frame along the x-axis.
Solution:
The transformation matrix is

$$\Lambda^{\bar{\mu}}_{\mu} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The components of the transformed tensor are

$$\begin{aligned} T^{\bar{\mu}\bar{\nu}} &= \Lambda^{\bar{\mu}}_{\mu} \left[\Lambda^{\bar{\nu}}_{\nu} T^{\mu\nu} \right] \\ &= \Lambda^{\bar{\mu}}_{\mu} \left[\Lambda^{\bar{\nu}}_{0} T^{\mu 0} + \Lambda^{\bar{\nu}}_{1} T^{\mu 1} + \Lambda^{\bar{\nu}}_{2} T^{\mu 2} + \Lambda^{\bar{\nu}}_{3} T^{\mu 3} \right] \end{aligned}$$

Since the off diagonal elements of $T^{\mu\nu}$ are all zeros we get zeros for all j

$$T^{\bar{\mu}\bar{\nu}} = \Lambda_0^{\bar{\mu}} \left[\Lambda_0^{\bar{\nu}} T^{00}\right] + \Lambda_1^{\bar{\mu}} \left[\Lambda_1^{\bar{\nu}} T^{11}\right] + \Lambda_2^{\bar{\mu}} \left[\Lambda_2^{\bar{\nu}} T^{22}\right] + \Lambda_3^{\bar{\mu}} \left[\Lambda_3^{\bar{\nu}} T^{33}\right]$$

So we get the transformed tensor as

$$\begin{bmatrix} \gamma^2\rho + \gamma^2 v^2 P & \gamma^2 v\rho + \gamma^2 vP & 0 & 0 \\ \gamma^2 v\rho + \gamma^2 vP & \gamma^2 v^2 P + \gamma^2 \rho & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} = \begin{bmatrix} \gamma^2(\rho + v^2 P) & \gamma^2 v(\rho + P) & 0 & 0 \\ \gamma^2 v(\rho + P) & \gamma^2(v^2\rho + P) & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

This is the required transformed tensor. □

(b) Suppose the pressure is a fixed ratio to the density. Compute the stress energy tensor in the moving frame for i) P = 0 (dust), ii) P= 1/3ρ (radiation) iii) $P = -\rho$ (cosmological constant).
Solution:
for P = 0 we get

$$\begin{bmatrix} \gamma^2\rho & \gamma^2 v\rho & 0 & 0 \\ \gamma^2 v\rho & \gamma^2 v^2\rho & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for P = 1/3 ρ we get

$$\begin{bmatrix} \gamma^2(\rho + v^2\frac{1}{3}\rho) & \gamma^2 v(\frac{4}{3}\rho) & 0 & 0 \\ \gamma^2 v(\frac{4}{3}\rho) & \gamma^2(v^2\rho + \frac{1}{3}\rho) & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{bmatrix}$$

for $P = -\rho$ we get

$$\begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 \\ 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & -\rho \end{bmatrix}$$

These are the transformed tensor. □