PHYS 576: Particle Physics

Homework #4

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1. **(Griffith 7.4)** Show that $u^{(1)}$ $u^{(2)}$ are *orthogonal*, in a sense that $u^{(1)\dagger}u^{(2)} = 0$. Likewise, show that $u^{(3)}$ and $u^{(4)}$ are orthogonal. Are $u^{(1)}$ and $u^{(3)}$ orthogonal?
Solution:
The bispinors $u^{(1)}$ and $u^{(2)}$ are

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x - ip_y}{E+m} \end{pmatrix} \qquad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix}$$

Checking for orthogonality with $u^{(1)\dagger}u^{(2)}$ we get

$$\begin{aligned} u^{(1)\dagger}u^{(2)} &= \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_x - ip_y}{E+m} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix} \\ &= 0 + 0 + \frac{p_z(p_x - ip_y)}{(E+m)^2} - \frac{p_z(p_x - ip_y)}{(E+m)^2} \\ &= 0 \end{aligned}$$

Since the product $u^{(1)\dagger}u^{(2)} = 0$ the two bispinors are orthogonal. Similarly the bispinors $u^{(3)}$ and $u^{(4)}$ are

$$u^{(3)} = \begin{pmatrix} \frac{p_x + ip_y}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad u^{(4)} = -\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Checking for orthogonality with $u^{(3)\dagger}u^{(4)}$ we get

$$\begin{aligned} u^{(3)\dagger}u^{(4)} &= -\begin{pmatrix} \frac{p_x + ip_y}{E+m} & -\frac{p_z}{E+m} & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{p_z(p_x + ip_y)}{(E+m)^2} - \frac{p_z(p_x + ip_y)}{(E+m)^2} + 0 + 0 \\ &= 0 \end{aligned}$$

Since the product $u^{(3)\dagger}u^{(4))} = 0$ the two bispinors are orthogonal.

Now checking for the orthogonality of $u^{(1)}$ and $u^{(3)}$ we get

$$
\begin{aligned}
u^{(1)\dagger}u^{(3)} &= \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_x-ip_y}{E+m} \end{pmatrix} \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{p_x - ip_y}{E+m} + 0 + 0 + \frac{p_x - ip_y}{E+m} \\
&= \frac{2p_x}{E+m}
\end{aligned}
$$

Since the product $u^{(1)\dagger}u^{(3))} \neq 0$ the two bispinors are not orthogonal. ☐

2. **(Griffith 7.17)**

(a) Express $\gamma^\mu\gamma^\nu$ as a linear combination of $1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5$ and $\sigma^{\mu\nu}$.
Solution:
The quantity $\sigma^{\mu\nu}$ is defined s

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \tag{1}$$

Also we know from the anti-commutation relation of the gamma matrices by definition

$$
\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\
\implies \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\nu &= 2g^{\mu\nu}
\end{aligned} \tag{2}
$$

Adding (1) and (2) we get

$$
\begin{aligned}
2\gamma^\mu\gamma^\nu &= 2(g^{\mu\nu} - i\sigma^{\mu\nu}) \\
\gamma^\mu\gamma^\nu &= g^{\mu\nu} - i\sigma^{\mu\nu}
\end{aligned}
$$

Here $g^{\mu\nu}$ is the Mankowski metric and is completely composed of numbers $1, -1$ and 0. So this is the required expression. ☐

(b) Construct the matrices σ^{12}, σ and σ^{23} and relate them to Σ^1, Σ^2, and Σ^3.
Solution:
By definition

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \qquad \sigma^{12} = \frac{i}{2}\left(\gamma^1\gamma^2 - \gamma^2\gamma^1\right) \tag{3}$$

$$
\begin{aligned}
\left[\gamma^1, \gamma^2\right] &= \gamma^1\gamma^2 - \gamma^2\gamma^1 \\
&= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} - \begin{pmatrix} -\sigma_2\sigma_1 & 0 \\ 0 & -\sigma_2\sigma_1 \end{pmatrix} \\
&= \begin{pmatrix} [\sigma_2, \sigma_1] & 0 \\ 0 & [\sigma_2, \sigma_1] \end{pmatrix} = \begin{pmatrix} -2i\sigma_3 & 0 \\ 0 & -2i\sigma_3 \end{pmatrix}
\end{aligned}
$$

Thus

$$\sigma^{12} = \frac{i}{2}\left[\gamma^1, \gamma^2\right] = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \Sigma^3$$

Similarly

$$\sigma^{13} = \frac{i}{2}\left(\gamma^1\gamma^3 - \gamma^3\gamma^1\right)$$

$$\begin{aligned}
\left[\gamma^1, \gamma^3\right] &= \gamma^1\gamma^3 - \gamma^3\gamma^1 \\
&= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sigma_1\sigma_3 & 0 \\ 0 & -\sigma_1\sigma_3 \end{pmatrix} - \begin{pmatrix} -\sigma_3\sigma_1 & 0 \\ 0 & -\sigma_3\sigma_1 \end{pmatrix} \\
&= \begin{pmatrix} [\sigma_3, \sigma_1] & 0 \\ 0 & [\sigma_3, \sigma_1] \end{pmatrix} = \begin{pmatrix} 2i\sigma_2 & 0 \\ 0 & 2i\sigma_2 \end{pmatrix}
\end{aligned}$$

Thus

$$\sigma^{13} = \frac{i}{2}\left[\gamma^1, \gamma^3\right] = -\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} = -\Sigma^2$$

Similarly

$$\sigma^{23} = \frac{i}{2}\left(\gamma^2\gamma^3 - \gamma^3\gamma^2\right)$$

$$\begin{aligned}
\left[\gamma^2, \gamma^3\right] &= \gamma^2\gamma^3 - \gamma^3\gamma^2 \\
&= \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_2\sigma_3 \end{pmatrix} - \begin{pmatrix} -\sigma_3\sigma_2 & 0 \\ 0 & -\sigma_3\sigma_2 \end{pmatrix} \\
&= \begin{pmatrix} [\sigma_3, \sigma_2] & 0 \\ 0 & [\sigma_3, \sigma_2] \end{pmatrix} = \begin{pmatrix} -2i\sigma_2 & 0 \\ 0 & -2i\sigma_2 \end{pmatrix}
\end{aligned}$$

Thus

$$\sigma^{23} = \frac{i}{2}\left[\gamma^2, \gamma^3\right] = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} = \Sigma^1$$

Here the commutation relation for the Pauli matrices $[\sigma_i, \sigma_j] = \varepsilon_{ijk} 2i\sigma_k$ has been used. This gives us the required relationship. □

3. **(Griffith 11.4)** As it stands Dirac Lagrangian treats ψ and $\bar{\psi}$ asymmetrically. Some people prefer to deal with them on an equal footing, using the modified Lagrangian

$$\mathcal{L} = \frac{i\hbar c}{2}\left[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right] - (mc^2)\bar{\psi}\psi$$

Apply the Euler-Lagrange equations to this $\mathcal{L}$, and show that you get the Dirac equations and its adjoint.
Solution:
The Euler Lagrange equation is for the Lagrangian density $\mathcal{L}(\partial_\mu\phi_1, \partial_\mu\phi_2, ..., \phi_1, \phi_2, \ldots)$ is

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right) = \frac{\partial\mathcal{L}}{\partial\phi_i}$$

For this modified Lagrangian we get

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) &= \frac{\partial\mathcal{L}}{\partial\bar{\psi}} \\
\partial_\mu\left(\frac{i\hbar c}{2}\left[-\gamma^\mu\psi\right]\right) &= \frac{i\hbar c}{2}\left[\gamma^\mu\partial_\mu\psi\right] - mc^2\psi \\
\frac{i\hbar c}{2}\left[-\gamma^\mu\partial_\mu\psi\right] &= \frac{i\hbar c}{2}\left[\gamma^\mu\partial_\mu\psi\right] - mc^2\psi \\
i\hbar\left(\gamma^\mu\partial_\mu\psi\right) - mc\psi &= 0 \qquad (4)
\end{aligned}$$

Similarly we get the other one with $\bar{\psi}$

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) &= \frac{\partial\mathcal{L}}{\partial\psi} \\
\partial_\mu\left(\frac{i\hbar c}{2}\left[\bar{\psi}\gamma^\mu\right]\right) &= \frac{i\hbar c}{2}\left[-\partial_\mu\bar{\psi}\gamma^\mu\right] - mc^2\bar{\psi} \\
\frac{i\hbar c}{2}\left[\partial_\mu\bar{\psi}\gamma^\mu\right] &= \frac{i\hbar c}{2}\left[-\partial_\mu\bar{\psi}\gamma^\mu\right] - mc^2\bar{\psi} \\
i\hbar\left(\partial_\mu\bar{\psi}\gamma^\mu\right) + mc\bar{\psi} &= 0 \qquad (5)
\end{aligned}$$

We find out that (4) and (5) are the Dirac equations and its adjoint. Thus this Lagrangian also gives the same Dirac equations. □

4. **(Griffith 11.20)** Construct the Lagrangian for *ABC* theory.
Solution:
Since the *ABC* model of particles are each scalar particle with spin 0, in free form, each can be described with a Klein-Gordan Lagrangian. So we can obtain the total Lagrangian with free form part of Klein-Gordan and interaction term. The free from Lagrangian is for each particle,if we assume the scalar field ϕ_A, ϕ_B and ϕ_C respectively,

$$\begin{aligned}
\mathcal{L}_A &= \frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A - \frac{1}{2}m_A^2\phi_A^2 \\
\mathcal{L}_B &= \frac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \frac{1}{2}m_B^2\phi_B^2 \\
\mathcal{L}_C &= \frac{1}{2}\partial_\mu\phi_C\partial^\mu\phi_C - \frac{1}{2}m_C^2\phi_C^2
\end{aligned}$$

The interaction terms as in the model has the strength of $-ig$. So the interaction term is

$$\mathcal{L}_{\text{int}} = -ig\phi_A\phi_B\phi_C$$

So the final Lagrangian is

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A - \frac{1}{2}m_A^2\phi_A^2 + \frac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \frac{1}{2}m_B^2\phi_B^2 \\
+\frac{1}{2}\partial_\mu\phi_C\partial^\mu\phi_C - \frac{1}{2}m_C^2\phi_C^2 - ig\phi_A\phi_B\phi_C
\end{aligned}$$

This is the required Lagrangian densitu for the *ABC* toy model. □