PHYS 576: Particle Physics

Homework #3

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1. **(Griffith 6.6)** The π^0 is a composite object ($u\bar{u}$ and $d\bar{d}$), and so equation 6.23 does not really apply. But lets pretend that the π^0 is a true elementary particle and see how close we came. Unfortunately, we don't know the amplitude $\mathcal{M}$; however it must have the dimensions of mass times velocity, and there is only one mass and one velocity available. Moreover, the emission of each photon introduces a factor of $\sqrt{\alpha}$ (the fine structure constant) into $\mathcal{M}$, so the amplitude must be proportional to α. On this basis, estimate the lifetime of π^0. Compare the experimental value.
Solution:
The decay rate for a particle decay is given by

$$\Gamma = \frac{S|\boldsymbol{p}|}{8\pi\hbar m_\pi^2}|\mathcal{M}|^2$$

Assuming the decay amplitude is $\mathcal{M} = \alpha m_\pi$ we get

$$\Gamma = \frac{1}{2}\frac{\boldsymbol{p}}{8\pi\hbar m_\pi^2}\left(\alpha m_\pi\right)^2 = \frac{\alpha^2}{16\pi\hbar}|\boldsymbol{p}|$$

The threshold energy of each outgoing photon is

$$E_\gamma = \frac{1}{2}m_\pi$$

We can use the fact that for photon $\left|\boldsymbol{p}_\gamma\right| = E_\gamma$, so the outgoing momentum can be written as

$$\left|\boldsymbol{p}_\gamma\right| = \frac{1}{2}m_\pi$$

Using this in the decay rate expression

$$\Gamma = \frac{\alpha^2 m_\pi}{32\pi\hbar}$$

So the lifetime is given by the reciprocal of decay rate

$$\text{Lifetime}(\tau) = \frac{32\pi\hbar}{\alpha^2 m_\pi}$$

Substuting $\alpha = \frac{1}{137}$ and mass of πon is $135MeV$ we get

$$\tau = \frac{32\pi(6.58\times10^{-22})}{135\cdot\frac{1}{137^2}} = 9.2\times10^{-18}s$$

So the estimated lifetime is 8.4×10^{-18}s. The mean lifetime from the particle data group listing[1] is $(8.30\pm0.19)\times10^{-17}$s, which is off by about an order magnitude.

□

[1] `http://pdg.lbl.gov/2018/listings/rpp2018-list-pi-zero.pdf`

2. **(Griffith 6.8)** consider th case of elastic scattering, $A+B \to A+B$, in the lab frame, (B initially at rest) assuming the target is so heavy $m_B \gg E_A$ that its recoil is negligible. Determine the differential scattering cross section.
Solution:
In the CM frame, for two body scattering we have the differential scattering cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_A+E_B)^2}\frac{|\boldsymbol{p}_f|}{|\boldsymbol{p}_i|}$$

Since the target particle is very heavy, it is essentially at rest after the scattering. So the expression is same for CM frame and lab frame. The energy and momentum of the incoming particle and outgoing particle is essentially the same as the target particle doesn't take any appreciable energy.

$$|\boldsymbol{p}_f| = |\boldsymbol{p}_i|$$

For the heavy particle, since it is essentially at rest, the energy is given by

$$E_B^2 = |\boldsymbol{p}_B|^2 + m_B^2$$

Which for $|\boldsymbol{p}_B| \approx 0$ gives

$$E_B = m_B$$

Since given that $E_A \ll m_B$ we can approximate

$$E_A + E_B = E_A + m_B \approx m_B$$

Also the particles are not identical so the factor $S = 1$ Thus the final expression for the scattering is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{m_B^2}$$

This is the required expression for the differential cross section of recoil. □

3. **(Griffith 6.9)** Consider the collision $1+2 \to 3+4$ in the lab frame (2 at rest), with particles 3 and 4 massless. Obtain the formula for differential cross section.
Solution:
The expression for the differential cross section for the collision $1+2 \to 3+4+\ldots+n$ is given by[2]

$$d\sigma = \frac{S\hbar^2}{4\sqrt{(p_1\cdot p_2)^2-(m_1m_2)^2}}|\mathcal{M}|^2(2\pi)^4\delta^4\left(p_1+p_2-p_3-\ldots-p_n\right)\times\prod_{i=3}^{n}\frac{1}{2\sqrt{p_i^2+m_i^2}}\frac{d^3p_i}{(2\pi)^3}$$

In the lab frame, with particle 2 at rest , we have

$$p_2^2 = |\boldsymbol{p}_2|^2 + m_2^2 = m_2^2$$

Also for the expression under the square root is ,

$$p_1 = (E_1, \boldsymbol{p}_1) \qquad p_2 = (m_2, 0), \Rightarrow p_1\cdot p_2 = E_1 m_2$$

This gives

$$\sqrt{(p_1\cdot p_2)^2-(m_1m_2)^2} = \sqrt{E_1^2m_2^2 - m_1^2m_2^2} = \sqrt{m_2^2(E_1^2-m_1^2)} = m_2|\boldsymbol{p}_1|$$

Substuting these for $n=4$ we get,

$$d\sigma = \frac{S\hbar^2}{4p_1m_2}\left(\frac{1}{4\pi}\right)^2\int|\mathcal{M}|^2\delta\left(E_1+E_2-|p_3|-|p_4|\right)\times\delta^3\left(p_1+p_2-p_3-p_4\right)\frac{d^3p_3d^3p_4}{|p_3||p_4|}$$

[2] Griffith eq. 6.38

the delta function make the p_4 integral trivial as

$$d\sigma = \frac{S\hbar^2}{64\pi^2|\boldsymbol{p}_1|m_2}\int |\mathcal{M}|^2\delta\left(E_1+m_2-|\boldsymbol{p}_3|-|\boldsymbol{p}_1-\boldsymbol{p}_3|\right)\frac{d^3\boldsymbol{p}_3}{|\boldsymbol{p}_3||\boldsymbol{p}_1-\boldsymbol{p}_3|}$$

Assumming 3 particle scatters off at an angle θ relative to the incident particle 1 we get

$$(\boldsymbol{p}_1-\boldsymbol{p}_3)^2 = |\boldsymbol{p}_1|^2+|\boldsymbol{p}_3|^2-2|\boldsymbol{p}_1||\boldsymbol{p}_3|\cos\theta$$

Also the volume element in the phase space $d^3\boldsymbol{p}_3$ can be written as

$$d^3\boldsymbol{p}_3 = |\boldsymbol{p}_3|^2 d|\boldsymbol{p}_3|d\Omega$$

Whore Ω is the solid angle. This enables us to write

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{64\pi^2|\boldsymbol{p}_1|m_2}\int_0^\infty |\mathcal{M}|^2\delta\left(E_1+m_2-|\boldsymbol{p}_3|-\sqrt{|\boldsymbol{p}_1|^2+|\boldsymbol{p}_3|^2-2|\boldsymbol{p}_1||\boldsymbol{p}_3|\cos\theta}\right)\frac{|\boldsymbol{p}_3|^2 d|\boldsymbol{p}_3|}{|\boldsymbol{p}_3|\sqrt{|\boldsymbol{p}_1|^2+|\boldsymbol{p}_3|^2-2|\boldsymbol{p}_1||\boldsymbol{p}_3|\cos\theta}}$$

At this point all the momentum p are the spatial momentum vectors of each particles (not to confuse with earlier notation p_i to mean four momentum), writing for each $|\boldsymbol{p}_i| = p_i$ and we have this integral in p_3 where p_3 is independent of p_1

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{64\pi^2 p_1 m_2}\int_0^\infty |\mathcal{M}|^2\delta\left(E_1+m_2-p_3-\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}\right)\frac{p_3^2 dp_3}{p_3\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}$$

This is no easy integral to work out, but lets try, suppose $x = p_3+\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}$ Differentiating this with respect to p_3 we get

$$\frac{\mathrm{d}x}{\mathrm{d}p_3} = 1+\frac{p_3-p_1\cos\theta}{\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}} = \frac{\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}+p_3-p_1\cos\theta}{\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}} = \frac{x-p_1\cos\theta}{\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}$$

This gives

$$\frac{dp_3}{\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}} = \frac{dx}{x-p_1\cos\theta}$$

Using this in the integral we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{64\pi^2 p_1 m_2}\int_0^\infty |\mathcal{M}|^2\delta\left(E_1+m_2-x\right)\frac{p_3 dx}{x-p_1\cos\theta}$$

This integral however is trivial because of the delta function, as it only picks up the terms for $x = E_1+m_2$ thus we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{64\pi^2 p_1 m_2}|\mathcal{M}|^2\frac{p_3}{E_1+m_2-p_1\cos\theta}$$

This can be simplified to write

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar}{8\pi}\right)^2\frac{S|\mathcal{M}|^2 p_3}{m_2 p_1\left(E_1+m_2-p_1\cos\theta\right)}$$

This is the expression of the scattering cross section. □