PHYS 431: Galactic Astrophysics

Homework #6

Prakash Gautam

December 11, 2017

1. Assuming the rotation curve for milky way is flat and$V(R) = R\Omega(R) = 200 km/s$ and $R_0 = 8kpc$. (a) Compute the Oort constants A and B, and the local epicyclic frequency κ. (b) If the Sun has v_x (radial) $= 10km/s$ and v_y (transverse) $= 5km/s$, calculate the Suns guiding radius R g and radial orbital amplitude X.
   Solution:
   For flat rotation curve $v(r) =$ constant. so, $\frac{dv}{dr} = 0$. .

$$A = \frac{1}{2}\frac{V(R)}{R_0} = \frac{1}{2}\frac{200km/s}{8kpc} = 12.50km/s/kpc$$
$$B = -\frac{1}{2}\frac{V(R)}{R_0} = -\frac{1}{2}\frac{200km/s}{8kpc} = -12.50km/s/kpc$$

   The value of κ is related to the oort constant as $\kappa^2 = -4B\Omega$

$$\Omega = V(R)/R = 200/8 = 2.5km/s/kpc; \qquad \kappa = \sqrt{4 * 12.50 * 2.5} = 11.18km/s/kpc$$

   Also

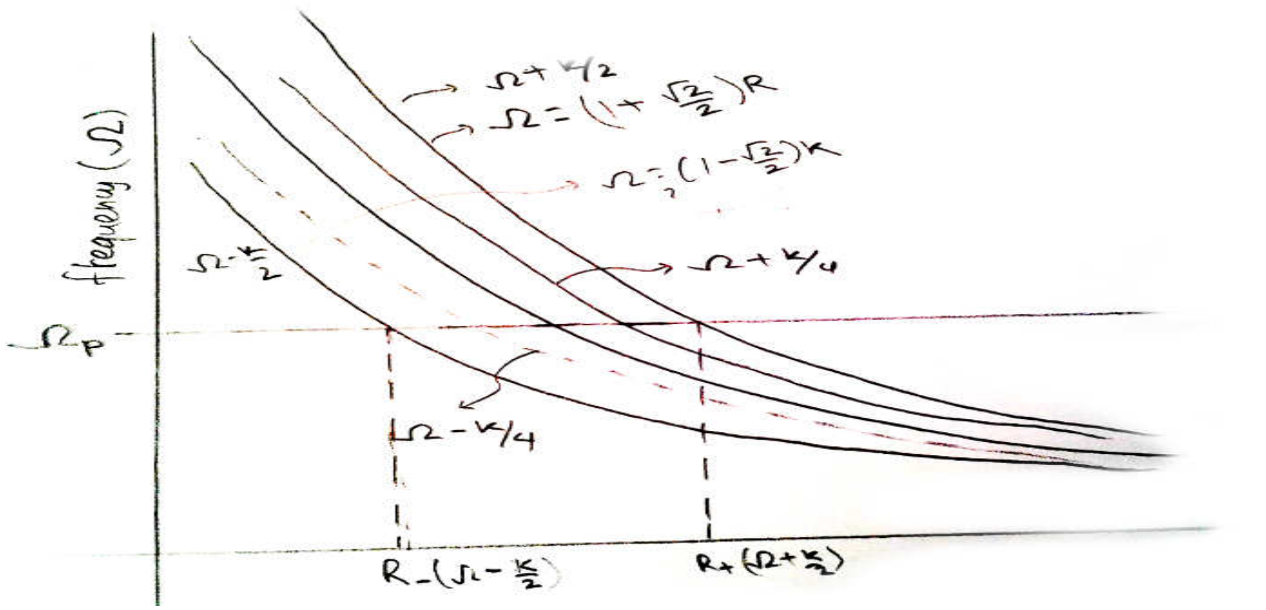
$$v_y = 2BX; \Rightarrow X = \frac{5km/s}{2 \cdot 12.50km/s/kpc} = 0.2kpc$$

   The guiding center is the sum of maximum displacement X and the closest approach so $R_g = R_0 + X = 8kpc + 0.2kpc = 8.20kpc$. ■

2. Show that, if the rotation curve of the Milky Way is flat near the Sun, then $\kappa = \sqrt{2}\Omega(R)$, so that locally $\kappa \approx 36km/s/kpc$. Sketch the curves ofΩ, $\Omega \pm \kappa/2$, and$\Omega \pm \kappa/4$ in a disk where $V(R)$ is constant everywhere, and show that the zone where two-armed spiral waves can persist is almost four times larger than that for four-armed spirals.
   Solution:
   For a flat rotation curve $V(R)$ =constant, so, $\frac{dV(R)}{dR} = 0$. The oort constant B is $B = -\frac{1}{2}\frac{V(R)}{R} = -\frac{\Omega}{2}$. But $\kappa^2 = -4B\Omega$. This gives

$$\kappa = \sqrt{-4 \cdot -\frac{\Omega}{2} \cdot \Omega} = \sqrt{2}\Omega(R)$$

   This gives the epicyclic frequency of the sun. The graph for $\Omega \pm \frac{\kappa}{2}$ and $\Omega \pm \frac{\kappa}{4}$ are The lowest and



highest values of R can be found at the points where Ω crosses the pattern speed Ω_p. The point $\Omega \pm \frac{\kappa}{2}$ crosses Ω_p are $R_{max} = (1 \pm \frac{1}{\sqrt{2}})R$ This gives the ration of region as

$$\frac{R_{max}}{R_{min}} = \frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = 5.8$$

Similarly The point $\Omega \pm \frac{\kappa}{4}$ crosses Ω_p are $R_{max} = (1 \pm \frac{1}{2\sqrt{2}})R$ This gives the ration of region as

$$\frac{R_{max}}{R_{min}} = \frac{1+\frac{1}{2\sqrt{2}}}{1-\frac{1}{2\sqrt{2}}} = 2.09$$

The region are approximately at the ratio of 3.0 ■

3. (a) Given the dispersion relation for a gas disk, $(\omega - m\Omega)^2 = k^2 v_s^2 - 2\pi G\Sigma|k| + \kappa^2$, Show that the group velocity is

$$v_g \equiv \left.\frac{\partial \omega}{\partial k}\right|_R = \text{sign}(k)\frac{|k|v_s^2 - \pi G\Sigma}{\omega - m\Omega}.$$

Solution:
Differentiating both sides of the given dispersion relation with respect to k, gives

$$2(\omega - m\Omega)\frac{\partial \omega}{\partial k} = 2kv_s^2 - 2\pi G\Sigma \text{sign}(k)$$

For any real number k we can write $k = |k|\text{sign}(k)$ using this in above expression can be rearranged in the form

$$\frac{\partial \omega}{\partial k} = \frac{2|k|\text{sign}(k)v_s^2 - 2\pi G\Sigma\ \text{sign}(k)}{2(\omega - m\Omega)} = \text{sign}(k)\frac{|k|v_s^2 - 2\pi G\Sigma}{\omega - m\Omega}$$

This gives the required group velocity as required. ■

(b) Show that, for a mirginally stable disk with $Q = \dfrac{v_s \kappa}{\pi G \Sigma} = 1$ the group velocity is equal to the sound speed v_s

Solution:

For $Q = 1$ we have $\pi G \Sigma = v_s \kappa$. Using this in the expression of group velocity gives

$$v_g = \text{sign}(k) \frac{|k| v_s^2 - v_s \kappa}{\omega - m\Omega}$$

We can use $\kappa = m\Omega$ and $k = \frac{\omega}{v_s}$. If we disregard the sign of k (ie, assume k as positive) the above expression becomes

$$v_g = \text{sign}(k) \frac{\frac{\omega}{v_s} v_s - m\Omega}{\omega - m\Omega} v_s = \frac{\omega - m\Omega}{\omega - m\Omega} v_s = v_s$$

This shows that the group velocity is (within a signof k) equal to the sound speed. ■

4. A satellite galaxy of mass M_s moves in a circular orbit of radius R in a spherically symmetric galactic halo of density $\rho(r) = v_c^2/4\pi G r^2$, with $M_s \ll v_c^2 R/G$. The stars (and dark matter particles) in the parent galaxy all have masses much less than M_s .

(a) Use the equation for dynamical friction to write down the drag force on the satellite as it orbits.

Solution:

The dynamical friction is given by,

$$-\frac{dv}{dt} = \frac{4\pi G (M_s + m)}{v^2} nm \ln(\Lambda);$$

For a satellite galaxy of mass M_s obiting at v_c the passing velocity is $V = v_c$ the drag force is $-M_s \frac{dv_c}{dt}$. Noting that for the galactic halo $nm = \rho(r)$ leads to.

$$F_{drg} = -M_s \frac{4\pi G^2 \cancelto{M_s}{(M_s + m)}}{v_c^2} \cdot \frac{v_c^2}{4\pi G r^2} \ln(\Lambda) = -\frac{M_s^2 G}{r^2} \ln(\Lambda)$$

This gives the expression for the drag force on the orbiting galaxy in the halo. ■

(b) The satellite sinks inward so slowly that it can be thought of as moving through a series of circular orbits, so its orbital speed at any radius r is always equal to the circular orbital speed at r. What is the angular momentum $L(r)$ of the satellite at radius r?

Solution:

The instantaneous speed at a distance r from the center is v_c, so the momentum is $P = M_s v_c$. The angular momentum is $L = r \times p$

$$L = r \times P = r M_s v_c \tag{1}$$

So the angular momentum of the galaxy at distance r is $M_s v_c r$ ■

(c) By equating the rate of change of L to the torque exerted on the satellite by dynamical friction, show that the distance $r(t)$ from the satellite to the center of the galaxy obeys the differential equation

$$\frac{dr}{dt} = -\frac{G M_s \ln(\Lambda)}{v_c r}$$

Solution:
The torque about the center of the galactic halo which the galaxy is orbiting is $\tau = F_{drg} r$, but $\tau = \frac{dL}{dt}$, combining these two give

$$\frac{dL}{dt} = F_{drg} r; \Rightarrow M_s v_c \frac{dr}{dt} = -\frac{M_s^2 G}{r^2} \ln(\Lambda) \cdot r; \Rightarrow \frac{dr}{dt} = -\frac{G M_s \ln(\Lambda)}{v_c r}$$

Which the required differential equation for the rate of change of distance of orbiting galaxy to center of halo. ■

(d) Solve this equation to estimate the time taken for the satellite to sink to the center of the parent galaxy.
Solution:
The time to fall t_f into the center of halo is given by the time for the distance of R_0 to 0 at the center of halo. Rearranging the above differential equation we get.

$$rdr = -\frac{G M_s \ln(\Lambda)}{v_c} dt; \Rightarrow \int_{R_0}^{0} rdr = -\int_{0}^{t_f} \frac{G M_s \ln(\Lambda)}{v_c} dt; \Rightarrow -\frac{{R_0}^2}{2} = \frac{G M_s \ln(\Lambda)}{v_c} t_f$$

So the time to sink is $t_f = \frac{R_0^2 v_c}{2 G M_s \ln(\Lambda)}$. ■

(e) Evaluate this time for a hypothetical "Magellanic Cloud" with $M_s = 2 \times 10^{10} M_\odot$ on an initially circular orbit of radius $R = 50kpc$ around our Galaxy, with $v_c = 220km/s$. Take $\Lambda = 20$.
Solution:
Substuting these values in the above expression

$$t_f = \frac{(50 \times 10^3)^2 \cdot 220 \times 10^3}{2 \cdot 6.67 \times 10^{-11} \cdot 2 \times 10^{10} M_\odot} = 3.28 \times 10^{16} s = 1.04 \times 10^9 yr = 1.04 Gyr$$

So the sink time of the cloud is $1.04Gyr$ ■

5. If the effective radius of the satellite galaxy in the previous problem is $R_s = 1.5kpc$, estimate the distance from the center of the parent galaxy at which tidal (differential) gravitational forces would significantly affect the satellites structure.
Solution:
The distance scale is given by

$$r_t = \left(\frac{M}{M_s}\right)^{\frac{1}{3}} R_s$$

Assuming $M_s = 2 \times 10^{10} M_\odot$ froom previous problem and the mass of galaxy to be that of Milky way $M = 5.8 \times 10^{11} M_\odot$

$$r_t = \left(\frac{5.8 \times 10^{11} M_\odot}{2 \times 10^{10} M_\odot}\right)^{\frac{1}{3}} 1500pc = 488.2pc$$

So the distance for significant effect is $488.2pc$ ■