PHYS 431: Galactic Astrophysics

Homework #5

Prakash Gautam

March 22, 2018

1. (a) Calculate the total gravitational potential energies of (i) a homogeneous sphere of mass M and radius a, and (ii) a Plummer sphere of mass M and scale length a

Solution:

The potential energy is $U = \frac{GM(r)m}{r}$. Where $M(r)$ is the mass inside of spherical shell of radius r. For a homogenous spherical distribution of ρ the $M(r) = \frac{4}{3}\pi r^3\rho$ and the additional mass increase due to increase in the radius of mass is $dm = \rho 4\pi r^2 dr$.If we bring dm from infinity to r then the increase in potential energy is

$$dU = \frac{GM(r)}{r}dm = \frac{GM(r)}{r}\cdot \rho 4\pi r^2 dr = \frac{G\frac{4}{3}\pi r^3\rho}{r}\cdot \rho 4\pi r^2 dr \tag{1}$$

The total potential energy is obtained by integrating Eq. (1) from 0 to the radius of the final sphere a.

$$U = \int_0^a \frac{16}{3}\pi^2\rho^2 G r^4 dr = \frac{16}{3}\pi^2 G\rho^2\frac{a^5}{5} = \frac{16\pi^2 G a^5}{15}\rho^2 \tag{2}$$

But for a homogenous sphere of radius a the density is $\rho = \frac{3M}{4\pi a^3}$. Using this is Eq. (2) we get

$$U = \frac{16}{3}\pi^2 G\frac{a^5}{5}\left(\frac{3M}{4\pi a^3}\right)^2 = \frac{3GM^2}{5a}$$

So the gravitational potential energy of homogenous shpere of mass M and radius a is $\frac{3GM^2}{5a}$. ■

Given any potential function we can always calculate the density function using the poisson equation.

$$\Phi = \frac{GM}{\sqrt{r^2+a^2}} \quad \text{Plummer Potential} \tag{3}$$

$$\nabla^2\Phi = 4\pi G\rho(r) \quad \text{Poisson's equation} \tag{4}$$

For spherical system the Laplacian operator is $\nabla^2 := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$. Calculating $\frac{\partial\phi}{\partial r}$ we have.

$$\frac{\partial\phi}{\partial r} = \frac{\partial}{\partial r}\left(\frac{GM}{\sqrt{r^2+a^2}}\right) = -\frac{GMr}{(r^2+a^2)^{3/2}}; \Rightarrow r^2\frac{\partial\phi}{\partial r} = -\frac{GMr^3}{(r^2+a^2)^{3/2}}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{GMr^3}{(r^2+a^2)^{3/2}}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[GM\left(1+\frac{a^2}{r^2}\right)^{-\frac{3}{2}}\right] = \frac{3GMa^2}{(r^2+a^2)^{5/2}}$$

Poisson's equation can be used to calculate the density function as $\rho(r) = \frac{\nabla^2\Phi}{4\pi G}$.

$$\rho(r) = \frac{1}{4\pi G} \cdot \frac{3GMa^2}{(r^2+a^2)^{5/2}} = \frac{3Ma^2}{4\pi} \frac{1}{(r^2+a^2)^{5/2}} \tag{5}$$

Eq.(5) gives the density function of the plummer model. This density function can be used to calculate the mass of spherical volume of radius r as:

$$M(r) = \int_0^r \rho(r) 4\pi r^2 dr = 4\pi \int_0^r \frac{3Ma^2}{4\pi} \frac{r^2}{(r^2+a^2)^{5/2}} dr = \frac{Mr^3}{(r^2+a^2)^{3/2}} \tag{6}$$

We can use Eq.(1) to calculate the potential energy equipped with the mass function and density function.

$$U = 4\pi G \int_0^\infty \frac{Mr^3}{(r^2+a^2)^{3/2}} \cdot r^2 \cdot \frac{3Ma^2}{4\pi} \frac{1}{(r^2+a^2)^{5/2}} dr = 3GM^2a^2 \int_0^\infty \frac{r^4}{(r^2+a^2)^4} = \frac{3\pi}{32} \frac{GM^2}{a}$$

So the total gravitational energy of plummer potential function is $\frac{3\pi}{32}\frac{GM^2}{a}$. ■

(b) Show that the total mass of the Plummer model is indeed M.

Solution:

Eq.(6) gives the mass cantained within the radius r for plummer sphere. The total mass of plummer sphere is the total mass contained inside the radius of $r = \infty$. Taking limit of Eq.(6) we get.

$$M_{tot} = \lim_{r\to\infty} \frac{Mr^3}{(r^2+a^2)^{3/2}} = \lim_{r\to\infty} \frac{M}{\left(1+\frac{a^2}{r^2}\right)^{\frac{3}{2}}} = M$$

This shows that the total mass of plummer model is M which appears in the potential function given by Eq. (3). ■

2. (a) Verify that the Kuzmin potential

$$\Phi_K(r,z) = -\frac{GM}{\sqrt{r^2+(a+|z|)^2}} \tag{7}$$

has $\nabla^2\Phi = 0$ for $z \neq 0$, and so represents a surface density distribution $\Sigma(r)$ in the plane $z = 0$.

Solution:

Writing $r^2 = x^2 + y^2$ where x and y are the cartesian coordinates corresponding to the r coordinate in cylindrical system. We get $\Phi = -GM(x^2+y^2+(a+|z|)^2)^{-1/2}$. In cartesian coordinate system $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. So each components of ths operator are.

$$\frac{\partial^2}{\partial x^2}\Phi = \frac{GM\left(2x^2-y^2-(a+z)^2\right)}{\left(x^2+y^2+(a+z)^2\right)^{\frac{5}{2}}}; \qquad \frac{\partial^2}{\partial y^2}\Phi = \frac{GM\left(-x^2+2y^2-(a+z)^2\right)}{\left(x^2+y^2+(a+z)^2\right)^{\frac{5}{2}}}$$

Since the potential is function of $|z|$ and the derivative of $|z|$ dosent't exist at $z = 0$. We take left hand and right hand derivative for the z component. Using $|z| = +z$ for right and $|z| = -z$ for left derivative, We get.

$$\frac{\partial^2}{\partial z_+^2}\Phi = \frac{GM\left(-x^2 - y^2 + 2(a+z)^2\right)}{\left(x^2 + y^2 + (a+z)^2\right)^{\frac{5}{2}}} \quad \frac{\partial^2}{\partial z_-^2}\Phi = \frac{GM\left(-x^2 - y^2 + 2(a-z)^2\right)}{\left(x^2 + y^2 + (a-z)^2\right)^{\frac{5}{2}}}$$

In each of the cases the total sum

$$\nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_+^2}\right)\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_-^2}\right)\Phi = 0$$

By use of Poisson's equation $\rho(r) = 1/4\pi G\nabla^2\Phi$ we conclude the mass density is zero everywhere except (possibly?) at $z = 0$. ■

(b) Use Gausss law to determine $\Sigma(r)$.
Solution:
The gauss law for gravitational field says $\oint_S \vec{E}\cdot d\vec{A} = 4\pi G M_{encl}$ where S is any arbitrary closed surface and M_{encl} is the mass inside that surface. Now that we know that there is no mass except at infinite plane $z = 0$, we are certain that the Gravitational force field is completely along $\hat{z}$. The force field along $\hat{z}$ is given by $\vec{E} = \frac{\partial\Phi}{\partial z}$. Since the potential function is not smooth, we have two different values for this derivative on either side of the disc.

$$\vec{E}_+ = \frac{\partial\Phi}{\partial z_+}(-\hat{z}) = \frac{\partial}{\partial z_+}\left(\frac{GM}{\sqrt{r^2 + (a+z)^2}}\right)(-\hat{z}) = -\frac{GM(a+z)}{\left(r^2 + (a+z)^2\right)^{3/2}}(-\hat{z})$$

$$\vec{E}_- = \frac{\partial\Phi}{\partial z_-}\hat{z} = \frac{\partial}{\partial z_-}\left(\frac{GM}{\sqrt{r^2 + (a-z)^2}}\right)\hat{z} = \frac{GM(a-z)}{\left(r^2 + (a-z)^2\right)^{3/2}}$$

If we take a cylindrical gaussian surface for S with surface Area $A\hat{z}$, The total mass inside the cylinder is $M_{encl} = \Sigma \times A$ and the flux though the surface $\oint \vec{E}\cdot d\vec{A} = E_+A + E_-A$. But E_z is uniform so we can calculate $E_+ = \frac{\partial\Phi}{\partial z^+}\Big|_{z=0} = \frac{GMa}{(r^2+a^2)^{3/2}}$. And similarly for $E_- = \frac{GMa}{(r^2+a^2)^{3/2}}$ So,

$$4\pi G\Sigma \times A = \frac{GMa}{(r^2 + a^2)^{3/2}}A + \frac{GMa}{(r^2 + a^2)^{3/2}}A; \Rightarrow \Sigma = \frac{Ma}{2\pi(r^2 + a^2)^{3/2}}$$

So the surface mass density of the Kuzmin disk is $\Sigma(r) = \frac{Ma}{2\pi(r^2+a^2)^{3/2}}$. ■

(c) What is the circular orbit speed for a particle moving in the plane of the disk?
Solution:
For this potential the total mass inside the spherical shell of radius r is simply the surface density times the area of great circle, so $M(r) = \Sigma(r)\pi r^2$. The transverse speed for a circular orbit

$$v_c = \sqrt{\frac{GM(r)}{r}} = \sqrt{\frac{G\pi r^2}{r}\cdot\frac{Ma}{2\pi(r^2 + a^2)^{3/2}}} = \sqrt{\frac{GMar}{2(r^2 + a^2)^{3/2}}}$$

This gives the speed of particle in circular orbit for Kuzmin potential. ■

3. For stars moving vertically in Galactic disk, with energy $E_z = \Phi(R_0, z) + {}^1\!/_2 v_z^2$, suppose the distribution function is

$$f(z, v_z) = \frac{n_0}{\sqrt{2\pi\sigma^2}} e^{-E_z/\sigma^2}.$$

Find the density $n(z)$ and give it's value $n(0)$. To construct self consistent model let $\Phi(z) = \sigma^2\phi$, show that

$$2\frac{d^2\phi}{dy^2} = e^{-\phi}, \qquad \text{Where } y = \frac{z}{z_0} \qquad \text{and} \qquad z_0^2 = \frac{\sigma^2}{8\pi G m n_0}$$

Solve this for $\phi(y)$ and hence find $\Phi(z)$ and $n(z)$. What is the value at large $|z|$?

Solution:

The number density is the zeroth moment of this distribution function so

$$n(z) = \int_{-\infty}^{\infty} f(z, v_z) dv_z = \frac{n_0}{\sqrt{2\pi\sigma^2}} 2\int_0^{\infty} e^{(-\Phi - {}^1\!/_2 v_z^2)/\sigma^2} = \frac{2n_0 e^{-\Phi/\sigma^2}}{\sqrt{2\pi\sigma^2}} \underbrace{\int_0^{\infty} e^{-\frac{v_z^2}{2\sigma^2}}}_{\sigma\sqrt{\frac{\pi}{2}}} = n_0 e^{-\Phi(R,z)/\sigma^2}$$

This gives the expression for $n(z)$. Since $\Phi(z = 0) = 0$ is given. $n(0) = n_0 e^{-\Phi(z=0)/\sigma^2} = n_0 e^0 = n_0$.
The total mass density $4\pi G\rho(z) = \nabla^2\Phi$. But $\rho(z) = mn(z)$ where m is the average mass. But for motion only along z, we can write $\nabla^2\Phi \equiv \frac{d^2\Phi}{dz^2}$. Also operator $dz^2 = z_0^2 dy^2$;. By poissons equation,

$$\frac{d^2\Phi}{dz^2} = 4\pi G m n(z); \quad \sigma^2 \frac{d^2\phi}{z_0^2 dy^2} = 4\pi G m n_0 e^{-\phi}; \quad \Rightarrow 2\frac{d^2\phi}{dy^2} = e^{-\phi(y)} \text{ if } z_0^2 = \frac{\sigma^2}{8\pi G m n_0}$$

Now solving this for ϕ as a function of y

$$2\frac{d^2\phi}{dy^2} = e^{-\phi(y)}$$

This differential equation should give a function $\phi(y)$ such that $n(z) = n_0 e^{-\phi(z_0 y)/\sigma^2} = n_0 sech^2(z/(2z_0))$ but I couldn't find any reasonable solution

For large value of $|z|$

$$\lim_{z_+ \to \infty} n_0 sech^2\left(\frac{z}{2z_0}\right) = 0; \qquad \lim_{z_- \to \infty} n_0 sech^2\left(-\frac{z}{2z_0}\right) = 0$$

So for large value of $|z|$ the density is zero. ■

4. A stellar system in which all particles are on radial orbits is described by the distribution function

$$f(\mathcal{E}, L) = \begin{cases} A\delta(L)(\mathcal{E} - \mathcal{E}_0)^{-1/2} & \text{if } \mathcal{E} > \mathcal{E}_0 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{E} = \psi - 1/2 v_t^2$is relative energy and $\mathcal{E}_0$ and A are constants.

(a) By writing $v^2 = v_r^2 + v_t^2$, where v_r and v_t are the radial and transverse velocities, and $L = rv_t$, prove that the volume element $d^3v = 2\pi v_t dv_t dv_r$ may be written $d^3v = \frac{\pi d\mathcal{E} dX}{r^2 v_r}$ where $X = L^2$.

(b) Hence show that the density is

$$\rho(r) = \begin{cases} Br^{-2} & \text{if } (r < r_0) \\ 0 & \text{if } (r \geq r_0) \end{cases}$$

where B is a constant and the relative potential at r_0 satisfies $\psi(r_0) = \mathcal{E}_0$.

Solution:

The number density is the zeroth moment of distribution function with respect to velocity. So

$$n(z) = \int f d^3v = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} A\delta(L)(\mathcal{E}-\mathcal{E}_0)^{-1/2} d^3v = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} A\delta(\sqrt{X})(\mathcal{E}-\mathcal{E}_0)^{-1/2}\frac{\pi d\mathcal{E}dX}{r^2 v_r}$$

$$= \int\limits_{-\infty}^{\infty} A\left(\int\limits_{-\infty}^{\infty}\delta(\sqrt{X})dX\right)(\mathcal{E}-\mathcal{E}_0)^{-1/2}\frac{d\mathcal{E}}{r^2 v_r}$$

If $r_0 < r$ then $\mathcal{E} = \psi(r) > \mathcal{E}_0$

$$\rho(r) = mn(r) = \int_{\mathcal{E}_0}^{\infty}\frac{mA(\mathcal{E}-\mathcal{E}_0)^{-1/2}}{r^2 v_r}d\mathcal{E}$$

$$= \frac{1}{r^2}\left[\frac{-2m(\mathcal{E}-\mathcal{E}_0)^{-3/2}}{3v_r}\right]_{\mathcal{E}_0}^{\infty} = Br^{-2}$$

But if $r_0 < r$ then $\mathcal{E} = \psi(r) < \mathcal{E}_0$ and $f(\mathcal{E}, L) = 0$ then,

$$\rho(r) = \int 0 d^3v = 0$$

This is a power law density with density decaying as square of the distance for a finite spherical region in space. ■