PHYS 431: Galactic Astrophysics

Homework #4

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1. Estimate the masses of star clusters having

(a) root mean square velocity 10 km/s and half-mass radius 10 pc,

Solution:

Given $v_{rms} = 10km/s$, the mean square speed is $\left\langle v^2 \right\rangle = (10km/s)^2 = 1 \times 10^4$ The total mass is given by

$$M = \frac{6R_h \left\langle v^2 \right\rangle}{G} = \frac{6 \cdot 1 \times 10^4 \cdot 10 \times 3.08 \times 10^{16}}{6.67 \times 10^{-11}} = 2.78 \times 10^{36} kg = 1.39 \times 10^6 M_\odot$$

So the mass of the cluster is $1.39 \times 10^6 M_\odot$ □

(b) mean density $100pc^{-3}$, rms velocity $2km/s$, and mean stellar mass $0.8M_\odot$,

Solution:

If the number density is n and average stellar mass is $\bar{m}$ then the mean mass density

$$\rho = n \cdot \bar{m} = 100pc^{-3} \cdot 0.8M_\odot = 80M_\odot/pc^3; v_{rms} = 2km/s \Rightarrow \left\langle v^2 \right\rangle = 4 \times 10^4$$

. The density volume relation $\rho = \frac{3M}{4\pi R^3} \Rightarrow R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$.

$$M = \frac{6R \left\langle v^2 \right\rangle}{G} = \frac{6 \left\langle v^2 \right\rangle}{G} \left(\frac{3M}{4\pi\rho}\right)^{1/3} \Rightarrow M = \left(\frac{6 \left\langle v^2 \right\rangle}{G} \left(\frac{3}{4\pi\rho}\right)^{\frac{1}{3}}\right)^{\frac{3}{2}} = 4.53 \times 10^{34} kg = 2.27 \times 10^4 M_\odot$$

□

(c) dynamical time 1×10^6yr and radius 1 pc.

Solution:

The dynamical time $\tau = \left(\frac{3\pi}{G\rho}\right)^{\frac{1}{2}}$. Using $\rho = \frac{3M}{4\pi R^3}$ we get

$$M = \frac{4\pi^2 R^3}{G\tau^2} = 1.75 \times 10^{35} kg = 8.79 \times 10^4 M_\odot$$

□

2. Interstellar gas in many galaxies is in virial equilibrium with the stars, in that the rms speed of the gas particles is the same as the rms stellar speed. Consider a large elliptical galaxy with a virial radius of 100 kpc and a mass of $1 \times 10^{12} M_\odot$ solar masses. Calculate the rms stellar velocity using the virial theorem. Hence estimate the temperature of the interstellar gas, assuming that it

is composed entirely of hydrogen.
Solution:

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \left(\frac{GM}{6R} \right)^{\frac{1}{2}} = \left(\frac{6.67 \times 10^{-11} \cdot 1 \times 10^{12} \cdot 1.9 \times 10^{30}}{6 \cdot 1 \times 10^4 \cdot 3.08 \times 10^{16}} \right)^{\frac{1}{2}} = 2.68 \times 10^5 m/s = 268 km/s$$

The mass of hydrogen is $m_H = 1.67 \times 10^{-27} kg$. If all the interstellar mass was composed of hydrogen then the temperature would be given by reation

$$\frac{1}{2} m_H \left\langle v^2 \right\rangle = \frac{3}{2} kT \Rightarrow T = \frac{m_H \left\langle v^2 \right\rangle}{3k} = \frac{1.67 \times 10^{-23} \cdot (268 \times 10^3)^2}{3 \cdot 1.68 \times 10^{23}} = 2.86 \times 10^6 K$$

□

3. Assuming an average stellar mass of $0.5 M_\odot$ and $\Lambda = r_c / 1AU$, lookup table values and find the relaxation time t_r at the center of globular cluster 47 Tucanae. Show that the crossing time $t_{cross} \approx 2 r_c / \sigma_r \sim 1 \times 10^{-3} t_{relax}$
Solution:
The total number of stars in the cluster is given by

$$N = \frac{\text{Total Mass}}{\text{Mean Mass}} = \frac{800 M_\odot}{0.5 M_\odot} = 1600$$

The density of stars from table is $\rho = 10^{4.9} M_\odot / pc^3$. The dynamical time of the stars can be now calculated as

$$\tau = \left(\frac{GM}{r_c^3} \right)^{-\frac{1}{2}} = 3.09 \times 10^5 yr$$

.

Now the relaxation time

$$t_{relax} = \frac{N}{8.5 \ln(\Lambda)} \tau = \frac{1600}{8.5 \ln(r_c / 1AU)} 3.09 \times 10^5 = 4.89 \times 10^6 yr$$

The cross time is

$$\frac{t_{cross}}{t_{relax}} = \frac{2 r_c}{\sigma_r t_{relax}} = \frac{2 \cdot 0.7 pc}{1.1 \times 10^4 \cdot 4.89 \times 10^6} = 2.54 \times 10^{-2}$$

□

4. The velocities of stars in a stellar system are described by a three-dimensional Maxwellian distributionthat is,

$$f(v) = A v^2 e^{-mv^2/2kT}$$

Here, A is a normalization constant, m is the stellar mass, assumed constant, k is Boltzmanns constant, and T is the temperature of the system. Verify the mean stellar kinetic energy is $\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT$

Solution:
The normalization condition gives

$$\int_0^\infty f(v)dv = \int_0^\infty Av^2 e^{-mv^2/2kT} dv = 1$$

To carry out the integration lets make some change of variables

$$\frac{mv^2}{2kT} = x; \Rightarrow v = \sqrt{\frac{2kT}{m}x}; \quad dv = \frac{kT}{mv}dx, \qquad \text{As } v \to \{0, \infty\} \quad x \to \{0, \infty\}$$

Using these variable transformation, our normalization integral becomes.

$$A\int_0^\infty v^2 e^{-x}\frac{kT}{mv}dx = A\int_0^\infty \frac{kT}{m}\sqrt{\frac{2kT}{m}x}e^{-x}dx = A\sqrt{2}\left(\frac{kT}{m}\right)^{\frac{3}{2}}\int_0^\infty \sqrt{x}e^{-x}dx = 1$$

But by definition of gamma function $\Gamma(n) = \int\limits_0^\infty x^{n-1}e^{-x}$ we get. And $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

$$A\sqrt{2}\left(\frac{kT}{m}\right)^{\frac{3}{2}}\int_0^\infty x^{\frac{3}{2}-1}e^{-x}dx = A\sqrt{2}\left(\frac{kT}{m}\right)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right) = 1 \Rightarrow A = \frac{1}{\frac{1}{2}\sqrt{2\pi}\left(\frac{kT}{m}\right)^{\frac{3}{2}}}$$

The expectation value for the square of speed can be calculated as:

$$\langle v^2\rangle = \int_0^\infty v^2 f(v)dv = A\int_0^\infty v^4 e^{-mv^2/2kT}dv$$

Carrying out same transformations as above we get.

$$\begin{aligned}
\langle v^2\rangle &= A\int_0^\infty 2^{\frac{3}{2}}\left(\frac{kT}{m}\right)^{\frac{5}{2}}x^{\frac{3}{2}}e^{-x}dx \\
&= A2^{\frac{3}{2}}\left(\frac{kT}{m}\right)^{\frac{5}{2}}\int_0^\infty x^{\frac{5}{2}-1}e^{-x} = A2^{\frac{3}{2}}\left(\frac{kT}{m}\right)^{\frac{5}{2}}\Gamma\left(\frac{5}{2}\right) = A2^{\frac{3}{2}}\left(\frac{kT}{m}\right)^{\frac{5}{2}}\frac{3}{4}\sqrt{\pi} \\
&= \frac{1}{\frac{3}{2}\sqrt{2\pi}\left(\frac{kT}{m}\right)^{\frac{3}{2}}} \times 2^{\frac{3}{2}}\left(\frac{kT}{m}\right)^{\frac{5}{2}}\frac{3}{4}\sqrt{\pi} \\
&= \frac{3kT}{m} \\
\Rightarrow \frac{1}{2}m\langle v^2\rangle &= \frac{3}{2}kT
\end{aligned}$$

So the kinetic energy of each mass is $\frac{3}{2}kT$ if the velocity destribution of the ensemble of mass follow Maxwellian distribution function. □

5. Work out the details of the simple evaporative model discussed in class. Stars evaporate from a cluster of mass M on a time scale $t_{ev} = \alpha t_R$, where $\alpha \gg 1$,so

$$\frac{dM}{dt} = -\frac{M}{\alpha t_R} \tag{1}$$

For pure evaporation, each escaping star carries off exactly zero energy (i.e. stars barely escape the cluster potential), so the total energy of the cluster remains constant.

(a) If the cluster potential energy can always be written as $U = -k\frac{GM^2}{2R}$ for fixed k, where R is a characterstic cluster radius, and assuming that the cluster is always in virial equilibrium, show that $R \propto M^2$ as the cluster evolves.

Solution:
The potential energy relation can be reorganized as

$$R = -\frac{kG}{2U}M^2; \qquad \Rightarrow R = \beta_0 M^2; \quad \Rightarrow R \propto M^2; \quad \text{Where } \beta_0 = -\frac{kG}{2U}$$

So $R \propto M^2$. □

(b) Assuming that the relaxation time t_R scales as $M^{1/2}R^{3/2}$ so

$$t_R = t_{R0}\left(\frac{M}{M_0}\right)^{1/2}\left(\frac{R}{R_0}\right)^{3/2} \tag{2}$$

Solve (**??**) to determine the lifetime of the cluster (in terms of its initial relaxation time t_{R0}). Also write down an expression for the mean cluster density as a function of time.

Solution:
We can write Eq. (**??**) as $t_R = \beta_1 M^{1/2}R^{3/2}$. Since $R = \beta_0 M^2$. We now have, $t_R = \beta_1 M^{1/2}(\beta_0 M^2)^{3/2}$;

$$\Rightarrow t_R = \beta_3 M^{7/2}$$

Suppose T is the lifetime of the cluster that had initial mass of M_i then as time goes from 0 to T mass goes from M_i to 0. Using t_R in Eq. (**??**) we get

$$\frac{dM}{dt} = -\frac{1}{\alpha}\frac{M}{\beta_3 M^{7/2}}; \Rightarrow \int_{M_i}^{0} M^{5/2}dM = -\beta_4 \int_0^T dt; \Rightarrow -\frac{2}{7}M_i^{7/2} = -\beta_4(T) \Rightarrow T \propto M_i^{7/2}$$

So the lifetime of the cluster is $T \propto M_i^{7/2}$ ■.

Now the density $\rho \propto \frac{M}{R^3}$. But for a system in dynamical equilibrium we have $R \propto M^2$. This gives $\rho \propto \frac{M}{(M^2)^3} = M^5 \Rightarrow M \propto \rho^{-5}$ Eq. (**??**) can be solved as a function of time as above and written as

$$M = \beta_5 t^{2/7} \Rightarrow M^{-5} = \beta_5 t^{-10/7} \Rightarrow \rho = \beta_6 t^{-10/7}$$

□

(c) Estimate this for a globular cluster of mass $5 \times 10^5 M_\odot$ radius $10pc$ and mean stellar mass $0.5 M_\odot$

Solution:

The density of this cluster is

$$\rho \approx \frac{M}{R^3} = \frac{5 \times 10^5 M_\odot}{10^3 pc^3} = 9.86 \times 10^{-14} kg/m^3 = 5 \times 10^2 M_\odot/pc^3$$

. The number of star is

$$N = \frac{M_{tot}}{m_{av}} = \frac{5 \times 10^5 M_\odot}{0.5 M_\odot} = 1 \times 10^6$$

The time scale then is

$$t = \left(\frac{GM}{R^3}\right)^{-1/2} = \frac{6.67 \times 10^{-11} \cdot 5 \times 10^5 M_\odot}{10^3 pc^3} = 6.67 \times 10^5 yr$$

□