PHYS 531: Galactic Astrophysics

Homework #3

Prakash Gautam

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1. Neutral hydrogen atoms in the cool interstellar medium have number density $n_H \approx 1cm^{-3}$ and temperature T100 K.

(a) Show that the average speed $\bar{v}$ of these atoms, defined by $\frac{1}{2}m_H\bar{v}^2 = \frac{3}{2}kT$(where m_H is the mass of hydrogen atom and k is Boltzmann's constant), is

$$\bar{v} \approx 2kms^{-1}\left(\frac{T}{100K}\right)^{1/2}.$$

Given that the average speed $\bar{v}$ of these atoms, defined by $\frac{1}{2}m_H\bar{v}^2 = \frac{3}{2}kT$ It can be rearranged into

$$\bar{v} = \sqrt{\frac{3kT}{m_H}} = \sqrt{\frac{3\times k\times 100}{m_H}}\sqrt{\frac{T}{100K}} = \sqrt{\frac{3\times 1.38\times 10^{-23}\times 100}{1.67\times 10^{-27}}}\left(\frac{T}{100K}\right)^{1/2} = 1.57kms^{-1}\left(\frac{T}{100K}\right)^{1/2}$$

(b) Hence show that the typical atomic center-of-mass kinetic energy is much greater than the energy difference between the hyperfine states associated with the 21-cm radio line.

The mean time between collisions for atoms in this environment is a few thousand years, while the mean time for an excited atom to emit a $21cm$ photon is $\approx 1.1\times 10^7$. As a result, the populations of the lower and upper hyperfine states are determined entirely by collisional processes and the states are populated proportional to their statistical weights, so three-quarters of all hydrogen atoms are in the upper state.

The energy associated with 21cm line is

$$E = \frac{hc}{\lambda} = 9.485\times 10^{-25}J = 5.92\times 10^{-6}eV$$

The typical energy is 13.6eV which is much greater than the energy associated with 21cm line

(c) Calculate the total 21-cm luminosity of a galaxy containing a total of $5\times 10^9 M_\odot$ of neutral hydrogen.
The total number of neutral hydrogen is $N = \frac{5\times 10^9 M_\odot}{m_H} = 1.13\times 10^{57}$.
The rate of emission of photon is $f = (1.1\times 10^7)^{-1}/yr = 2.28\times 10^{-15}s^{-1}$.
So the total luminosity due to 21cm photon is given by

$$L = N\cdot f\cdot E = 3.11\times 10^{18}W$$

So the total luminosity fo the given galaxy is 3.11×10^{18}

2. What are the sound speed and Jeans mass. (In all cases, assume an adiabatic index $\gamma = \frac{5}{3}$)

(a) in a molecular cloud core (pure H_2) of temperature 10 K and number density 1×10^6 molecules/cm^3? Mass of hydrogen molecule H_2 is $m_{H_2} = 3.34 \times 10^{-27}$, $T = 10K$

$$C_s = \sqrt{\frac{\gamma kT}{m}} = 262.41 m/s; \quad \rho = m_{H_2} n = 3.34 \times 10^{-15} kgm^{-3}$$
$$\lambda_j = \sqrt{\frac{\pi c_s^2}{\rho}} = 8.048 \times 10^9 m; \quad M_j = \frac{4\pi}{3}\rho\lambda_j^2 = 7.29 \times 10^{15} kg$$

(b) in atomic hydrogen gas with temperature 100 K and number density 1 atom/cm^3 Mass of hydrogen atom is $m_H = 1.67 \times 10^{-27}$, $T = 100K$

$$C_s = \sqrt{\frac{\gamma kT}{m}} = 117.56 m/s; \quad \rho = m_H n = 1.67 \times 10^{-21} kgm^{-3}$$
$$\lambda_j = \sqrt{\frac{\pi c_s^2}{\rho}} = 5.09 \times 10^{13} m; \quad M_j = \frac{4\pi}{3}\rho\lambda_j^2 = 9.22 \times 10^{20} kg$$

(c) in hot ionized hydrogen with temperature 1×10^6 K and number density 1×10^{-3} protons/cm^3? Mass of ionized hydrogen molecule is $m_p = 1.67 \times 10^{-27}$, $T = 100K$

$$C_s = \sqrt{\frac{\gamma kT}{m}} = 1.17 \times 10^5 m/s; \quad \rho = m_p n = 1.67 \times 10^{-24} kgm^{-3}$$
$$\lambda_j = \sqrt{\frac{\pi c_s^2}{\rho}} = 1.67 \times 10^{17} m; \quad M_j = \frac{4\pi}{3}\rho\lambda_j^2 = 2.91 \times 10^{28} kg$$

3. Air at sea level on Earth has density = 1.2 kg/m3 and sound speed vs = 330 m/s.

(a) What is its Jeans length? What is the Jeans mass?

$$\lambda_j = \sqrt{\frac{\pi c_s^2}{\rho}} = 5.339 \times 10^2 m; \quad M_j = \frac{4\pi}{3}\rho\lambda_j^2 = 7.65 \times 10^8 kg$$

The Jeans length is $533.9m$ and the Jeans mass is $7.65 \times 10^8 kg$

(b) By how much does the self-gravity of air change the frequency of a sound wave of wavelength 1 m? The frequency of $1m$ wavelength wave on earth is $f = v_s/\lambda = 330Hz$ The change in frequency due to gravitation is related by

$$f^2 - f_n^2 = \frac{G\rho}{\pi} \tag{1}$$

If we supppose changed frequency $f_n = f + \Delta f$ and Δf is very small then

$$f^2 - f_n^2 = f^2 - (f + \Delta f)^2 = f^2 - f^2\left(1 + \frac{\Delta f}{f}\right)^2 \approx f^2 - f^2\left(1 + 2\frac{\Delta f}{f}\right) = 2\Delta f \cdot f$$

Substuting this difference into (**??**) we get

$$\Delta f = \frac{G\rho}{2\pi f} = \frac{6.672 \times 10^{-11} \cdot 1.2}{2\pi \cdot 330} = 3.86 \times 10^{-14} Hz$$

(4a) A particle is dropped (from radius a with zero velocity) into the gravitational potential corresponding to a static homogeneous sphere of radius a and density . Calculate how long the particle takes to reach the center of the sphere.
Let the density of the mass density of the homogenous sphere be ρ. Also le the mass of the sphere within the shell of radius r be $M(r)$.

$$M(r) = \rho V(r) = \rho \frac{4}{3}\pi r^3$$

Writing the equation of motion from Newton's laws.

$$\ddot{r} = -\frac{GM(r)}{r^2} = -\frac{G\rho\frac{4}{3}\pi r^3}{r^2} = -\frac{4}{3}G\pi\rho r = -\omega^2 r \quad \left(\text{where } w^2 = \frac{4}{3}G\pi\rho\right) \tag{2}$$

The second order differential equation (**??**) is the well known SHM equation which has periodic solution of the form.

$$r(t) = A\cos(\omega t) + B\sin(\omega t) \tag{3}$$

Differentiating (**??**) we get

$$\dot{r}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

where A and B are the parameters determined by the boundary value. Since the particle starts from the surface of the sphere $r(0) = a$ and the initial speed $\dot{r}(0) = 0$. Using these boundary values we find the values of A and B. The thus determined are $A = a$ and $B = 0$. So (**??**) becomes

$$r(t) = a\cos(\omega t) \qquad \text{Where } \omega = \sqrt{4/_3 G\pi\rho} \tag{4}$$

If T is the time the particle takes to reach the center of the spherical distribution then $r(T) = 0$ so the solution of (**??**) gives

$$\omega T = \frac{\pi}{2} \Rightarrow T = \frac{\pi}{2\omega} = \sqrt{\frac{3\pi}{16G\rho}}$$

The time T is the time the particle takes to reach the center of spherical distribution.

(b) Calculate the time required for a homogeneous sphere of radius a and density with no internal pressure support to collapse under its own gravity.
If the spherical distribution collapses by its own gravity, then as the particle on the surface is pulled inward towards the center, the mass compresses and so the mass inside the spherical shell at any time is constant

$$M(r) = \rho\frac{4}{3}\pi a^3$$

Writing the equation of motion from Newton's laws.

$$\ddot{r} = -\frac{GM(r)}{r^2} = -\frac{G\rho\frac{4}{3}\pi a^3}{r^2} = -\frac{\omega^2}{r^2} \quad \left(\text{where } w^2 = \frac{4}{3}G\pi\rho a^3\right) \tag{5}$$

We can transform $\ddot{r}$

$$\ddot{r} \equiv \frac{d\dot{r}}{dt} \equiv \frac{dr}{dt}\frac{d\dot{r}}{dr} \equiv \dot{r}\frac{d\dot{r}}{dr} \tag{6}$$

On using (**??**) (**??**) becomes

$$\dot{r}d\dot{r} = \left(-\frac{\omega^2}{r^2}\right)dr \qquad \Rightarrow \qquad \frac{1}{2}\dot{r}^2 = \frac{\omega^2}{r} + K$$

The boundary condition is that at $r = a$ the starting speed of particle is $\dot{r} = 0$ Substuting this back we find $K = {}^{-\omega^2}/_{a}$. We get

$$\dot{r} = \sqrt{2}\omega\sqrt{\frac{1}{r} - \frac{1}{a}} \qquad \Rightarrow \qquad \left(\frac{1}{r} - \frac{1}{a}\right)^{-\frac{1}{2}} dr = \sqrt{2}\omega dt \tag{7}$$

The solution of (??) is[1]

$$a^{\frac{3}{2}} \sin^{-1}\left(\sqrt{\frac{r}{a}}\right) - \frac{r-a}{\sqrt{\frac{1}{a} - \frac{1}{r}}} = \sqrt{2}\omega t + C \tag{8}$$

$$\lim_{r \to a} \frac{r-a}{\sqrt{\frac{1}{a} - \frac{1}{r}}} = 0; \qquad \lim_{r \to a} \sin^{-1}\left(\sqrt{\frac{r}{a}}\right) = \frac{\pi}{2} \quad \Rightarrow \quad C = \frac{\pi}{2}a^{3/2} \tag{9}$$

Using (??) in (??) we get

$$a^{\frac{3}{2}}\left(\sin^{-1}\left(\sqrt{\frac{r}{a}}\right) - \frac{\pi}{2}\right) - \frac{r-a}{\sqrt{\frac{1}{a} - \frac{1}{r}}} = \sqrt{2}\omega t \tag{10}$$

If T is the time the particle takes to reach the center $r(T) = 0$ so the solution of (??) gives

$$\sqrt{2}\omega T = \frac{\pi}{2}a^{\frac{3}{2}} \Rightarrow T = \frac{\pi}{2\sqrt{2}\omega}a^{3/2} = \sqrt{\frac{a^3\pi^2}{8 \cdot {}^4\!/_3 G\rho\pi a^3}} = \sqrt{\frac{3\pi}{32G\rho}}$$

The time T is the time the particle takes to reach the center of spherical distribution which is the time of the collapse of the mass distribution under its own gravitational pull.

[1]solved by Sympy 1.1.1 under python 3.5