PHYS 531: Galactic Astrophysics

Homework #2

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1. A certain telescope has limiting visual apparent magnitude $m_V = 22$. What is the maximum distance at which it could detect (a) the Sun (absolute magnitude $M_V = 4.8$), (b) an RR Lyrae variable ($M_V = 0.75$), a Cepheid variable ($M_V = 3.5$), and (d) a type Ia supernova ($M_V = 20$).
   Solution:
   If m_V is the limiting aparrant magnitude of the telescope, anything with apparant magnitude greater than m_V would not be resolved by the telescope. So the maximum distance that the telescope can still resolve is the distance in which the apparant magnitude of each of the stars is equal to the limiting apparant magnitude.

   If we suppose d_{max} is the maximum distance. Then

   $$M_v - m_v = -5\log\left(\frac{d_{max}}{10}\right)$$

   $$Rightarrow d_{max} = 10 \times 10^{\frac{m_v - M_v}{5}}$$

   Since the limiting magnitude $(m_v) = 22$

   $$d_{max} = 10 \times 10^{\frac{22-M_v}{5}} \text{in Parsec}$$

   - For Sun $M_V = 4.8$, limiting distance $d_{max} = 10 \times 10^{\frac{22-4.8}{5}} = 27.54 kpc$
   - For RR Lyre $M_V = 0.75$, limiting distance $d_{max} = 10 \times 10^{\frac{22-0.75}{5}} = 177.82 kpc$
   - For Cepheid variable $M_V = -3.5$, limiting distance $d_{max} = 10 \times 10^{\frac{22+3.5}{5}} = 1.25 Mpc$
   - For Ia Supernova $M_V = -20$, limiting distance $d_{max} = 10 \times 10^{\frac{22+20}{5}} = 2.511 Gpc$

   □

2. A simple axisymmetric model of the stellar number density $n(R, z)$ in the Galactic disk is

   $$n(R, z) = n_0 e^{-R/h_R} e^{-|z|/h_z},$$

   where R is distance from the Galactic center, z is distance from the disk plane, and h_R and h_z are (constant) scale heights. (a) If all stars have the same luminosity L_*, integrate the above expression with respect to z to determine the disk surface brightness $\Sigma(R)$ (that is, the total luminosity per unit area at any given location). (b) Now integrate Σ with respect to R to determine the total luminosity L_G of the Galaxy. (c) If $L_G = 2 \times 10^{10} L_\odot$, and $h_R = 4$ kpc, what is the local surface brightness in the vicinity of the Sun, at R = 8 kpc? (d) If $h_z = 250$ pc and $L_* = L_\odot$, calculate the local density of stars in the solar neighborhood (at z = 0).
   Solution:

Given all stars have same luminosity L_* the luminosity per unit area is:

$$\Sigma(R) = \int_{-\infty}^{\infty} L_* n_0 e^{-\frac{R}{h_R}} e^{-\frac{|z|}{h_z}} \, dz = 2 \int_{0}^{\infty} L_* n_0 e^{-\frac{R}{h_R}} e^{-\frac{z}{h_z}} \, dz$$
$$= -L_* h_z n_0 e^{-\frac{R}{h_R}} \left[e^{-\frac{z}{h_z}} \right]_0^{\infty} = 2 n_0 h_z e^{-\frac{R}{h_R}} L_*$$

Now for the total Luminosity the function $\Sigma(R)$ is integrated from $R = 0$ to ∞.

$$L_G = \int_0^{\infty} \Sigma(R) dR = \int_0^{\infty} 2 n_0 h_z e^{-\frac{R}{h_R}} L_* dR$$
$$= 2 n_0 h_z L_* \left(-h_r \left[e^{-\frac{R}{h_R}} \right]_0^{\infty} \right) = 2 n_0 h_R h_z L_*$$

The above expression for L_G gives the total luminosity of galaxy in terms of the luminosity of each stars L_*

For vicinity of sun at $R = 8kpc$ and $L_G = 2 \times 10^{10} L_\odot$ and $h_R = 4kpc$

$$L_G = 2 n_0 h_R h_z L_* \qquad \Rightarrow h_z = \frac{L_G}{2 L_* h_R n_0} \Rightarrow h_z = \frac{2.5 \times 10^9}{n_0 L_*} L_\odot$$

So the local surface brightness $\Sigma(R)$ at the vicinity of sun then is

$$\Sigma(R) = 2 n_0 h_z e^{-\frac{R}{h_R}} L_* = 2 n_0 \cdot \frac{2.5 \times 10^9}{n_0 L_*} \cdot e^{-\frac{8}{4}} L_* \Rightarrow 6.76 \times 10^8 \ L_\odot$$

The local density of stars around $z = 0$ is

$$n(8kpc, 0) = n_0 e^{\frac{8}{4}} e^0 = 0.13 n_0$$

□

3. (a) Given the definitions of the Oort constants A and B presented in class (Eqs. 2.13 and 2.16 in the text),

$$A = -\frac{1}{2} R \left(\frac{V}{R} \right)' \Bigg|_{R=R_0} \qquad\qquad B = -\frac{1}{2} \frac{(RV)'}{R} \Bigg|_{R=R_0}$$

verify that $A + B = V'(R_0)$ and $AB = V_0/R_0$, where V (R) is the Galactic rotation law, R_0 is the distance from the Sun to the Galactic center, and $V_0 = V(R0)$.
(b) Hence write down an estimate of V_0, if $R_0 = 8$kpc.
(c) Consider the spherically symmetric density distribution given by

$$\rho(R) = \rho_0 \left(1 + \frac{R^2}{a^2} \right)^{-1}$$

Derive an expression for the mass inside radius R. What is the circular orbital speed V (R) at radius R? Hence determine the form of A(R) and B(R) for $R \gg a$.
Solution:

$$A = -\frac{1}{2}R\left(\frac{V}{R}\right)'$$
$$= -\frac{1}{2}R\left(\frac{V'}{R} - \frac{V}{R^2}\right)$$
$$= -\frac{1}{2}V' + \frac{1}{2}\frac{V}{R}$$

Evaluating at $R = R_0$

$$A = -\frac{1}{2}V'(R_0) + \frac{1}{2}\frac{V(R_0)}{R_0}$$

$$B = -\frac{1}{2}\frac{(RV)'}{R}$$
$$= -\frac{1}{2}\frac{1}{R}(V + RV')$$
$$= -\frac{1}{2}\frac{V}{R} - \frac{1}{2}V'$$

Evaluating at $R = R_0$

$$B = -\frac{1}{2}\frac{V(R_0)}{R_0} - \frac{1}{2}V(R_0)'$$

Now that we have the values for each constants A and B.

$$A + B = -V'(R_0) \qquad A - B = \frac{V(R_0)}{R_0}$$

Let us conider a hollow shell of radius R with thickness dR. Then the volume of the differential shell is

$$dV = 4\pi R^2 dR$$

The differential relation for mass can be written as.

$$dM = \rho(R)dV$$
$$= \rho_0\left(1 + \frac{R^2}{a^2}\right)^{-1} 4\pi R^2 dR$$

The total mass enclosed in the sphere of radius R is given by the integral of dM from 0 to R

$$M(R) = \int_0^R dM = \int_0^R \rho_0\left(1 + \frac{R^2}{a^2}\right)^{-1} 4\pi R^2 dR$$
$$= 4\pi\rho_0 \int_0^R \frac{R^2}{1 + \frac{R^2}{a^2}} dR$$
$$= 4\pi\rho_0 a^2\left(R - a\tan^{-1}\left({}^{R}\!/_{a}\right)\right)$$

$$M(R) = 4\pi\rho_0 a^2\left(R - a\tan^{-1}\left({}^{R}\!/_{a}\right)\right) \tag{1}$$

To calculate the $V(R)$ we can use the relation.

$$\frac{V^2(R)}{R} = \frac{GM(R)}{R^2}$$
$$\frac{V^2(R)}{R} = \frac{G4\pi\rho_0 a^2\left(R - a\tan^{-1}\left({}^{R}\!/_{a}\right)\right)}{R^2} \qquad \text{Substuting M(R) from (1)}$$
$$V(R) = 2a\sqrt{G\pi\rho_0\left(1 - \frac{a}{R}\tan^{-1}\left(\frac{R}{a}\right)\right)}$$

if $R \gg a$ then $\tan^{-1}\left(\frac{R}{a}\right) \approx \frac{\pi}{2}$ also $\frac{a}{R} \to 0$ Then.

$$V(R) = 2a\sqrt{G\pi\rho_0}$$

Since V has no dependence on R, $V' = 0$

$$A(R) = -\frac{1}{2}V' + \frac{1}{2}\frac{V}{R} = 0 + \frac{1}{2}\frac{2a\sqrt{G\pi\rho_0}}{R} = \frac{a\sqrt{G\pi\rho_0}}{R}$$

$$B(R) = -\frac{1}{2}\frac{V}{R} - \frac{1}{2}V' = -\frac{1}{2}\frac{2a\sqrt{G\pi\rho_0}}{R} + 0 = -\frac{a\sqrt{G\pi\rho_0}}{R}$$

□

4. If our Galaxy has a flat rotation curve with $V_0 = 210$ km/s and the total luminosity of the disk is as in Problem 2, what is the Galactic mass to light ratio M/L inside (a) the solar circle ($R_0 = 8$ kpc), (b) $10R_0$? Compare these with the mass to light ratio of a Salpeter stellar mass distribution (see Homework 1, Problem 3) with $M_l = 0.2M_\odot$, $M_u = 100M_\odot$.
Solution:
Total luminosity inside of radius R can be calculated as

$$L(R) = \int_0^R \Sigma(R)dR = \int_0^R 2n_0h_ze^{-\frac{R}{h_R}}L_*$$
$$= 2h_Rh_zn_0\left(1 - e^{-\frac{R}{h_R}}L_*\right)$$

Substuting $h_r = 4kpc, h_z = 250pc, L_* = L_\odot$ in above expression

$$L(R) = 2 \times 10^6\left(1 - e^{\frac{R}{4kpc}}\right)n_0L_\odot$$

If the rotation curve is flat, the mass can be calculated as

$$M(R) = \frac{RV^2}{G} = 48.83R\,M_\odot/pc$$

For $R = 8kpc$

$$L(R) = 2.0 \times 10^6\left(1 - e^{-2}\right)n_oL_\odot = 1.72 \times 10^6 n_0L_\odot$$
$$M(R) = \frac{RV^2}{G} = 48.83 \times 8000M_\odot = 3.90 \times 10^5M_\odot$$

The ratio then is:

$$M/L = \frac{1.72 \times 10^6 n_0L_\odot}{3.90 \times 10^5M_\odot} = 0.22n_0{}^{M_\odot}/_{L_\odot}$$

For $R = 10R_0 = 80kpc$

$$L(R) = 2.0 \times 10^6\left(1 - e^{-20}\right)n_oL_\odot = 1.99n_0L_\odot$$
$$M(R) = \frac{RV^2}{G} = 48.83 \times 80000M_\odot = 3.90 \times 10^6M_\odot$$

The ratio then is:

$$M/L = \frac{1.99L_\odot}{3.90 \times 10^6M_\odot} = 5.1 \times 10^{-7}n_0{}^{M_\odot}/_{L_\odot}$$

For salpeter distribution $\xi(M) = AM^{-2.35}$ The total mass is

$$M = \int_{.2M_\odot}^{100M_\odot} M\xi(M)dM = \int_{.2M_\odot}^{100M_\odot} AM^{-1.35}dM = 1.49 \times 10^6A$$
$$L = \int_{.2M_\odot}^{100M_\odot} M^4\xi(M)dM = \int_{.2M_\odot}^{100M_\odot} AM^{1.65}dM = 2.13 \times 10^4A$$

The ratio is

$$M/L = \frac{1.49 \times 10^6A}{2.13 \times 10^4A} = 69.95$$

□