PHYS 431: Galactic Astrophysics

Homework #1

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1. Assume that the Galaxy is 10Gyr old, the rate of star formation in the past was proportional to $e^{\frac{-t}{T}}$ where t is the time since the galaxy formed and $T = 3$Gyr, and the stellar lifetimes are given by

$$t(M) = 10Gyr\left(\frac{M}{M_\odot}\right)^{-3}$$

Calculate the framctions of all (a) $2M_\odot$ and (b) $5M_\odot$ stars ever formed that are still around today.
Solution:
Let t_2 and t_5 be the lifetimes of $2M_\odot$ stars and $5M_\odot$ stars. Then

$$t_2 = 10\ times\left(\frac{2M_\odot}{M_\odot}\right)^{-3} = 1\frac{1}{4}Gyr = 1.25Gyr$$

$$t_5 = 10\ times\left(\frac{5M_\odot}{M_\odot}\right)^{-3} = \frac{2}{25}Gyr = 0.8Gyr$$

If N_{2f} is the total $2M_\odot$ stars ever formed, then

$$N_{2f} = \int_0^{10} ke^{-\frac{t}{T}}dt = -\frac{k}{T}\left[e^{-\frac{10}{3}} - 1\right] = -0.32k$$

Any $2M_\odot$ star formed earlier than t_2 from today are all gone so the remaining $2M_\odot$ stars are formed between $10 - t_2 = 10 - 1.25 = 8.75$Gyr and today (10Gyr) from the beginning.

$$N_{2r} = \int_{8.75}^{10} ke^{-\frac{t}{T}}dt = -\frac{k}{T}\left[e^{-\frac{10}{3}} - e^{-\frac{8.75}{3}}\right] = -6.14\times10^{-3}k$$

So the ratio of total $2M_\odot$ star still formed to that are still around is

$$\frac{N_{2r}}{N_{2f}} = \frac{-6.14\times10^{-3}k}{-0.32k} = 1.91\times10^{-2}$$

Since the star formation rate is independent of mass, the total $5M_\odot$ stars ever formed is equal to the total $2M_\odot$ stars. So, $N_{5f} = -0.321k$. Any $5M_\odot$ star formed earlier than t_5 from today are all gone so the remaining $5M_\odot$ stars are formed between $10 - t_2 = 10 - 0.08 = 9.92$Gyr and today (10Gyr) from the beginning.

$$N_{5r} = \int_{9.92}^{10} ke^{-\frac{t}{T}}dt = -\frac{k}{T}\left[e^{\frac{10}{3}} - e^{-\frac{9.92}{3}}\right] = -3.21\times10^{-4}k$$

So the ratio of total $5M_\odot$ star still formed to that are still around is

$$\frac{N_{5r}}{N_{5f}} = \frac{-3.21\times10^{-4}k}{-0.32k} = 9.99\times10^{-4}$$

$\square$

2. (a) A close (i.e. unresolved) binary consists of two stars each of apparent magnitude m. What is the apparent magnitude of the binary?
(b) A star has apparent magnitude $m_V = 10$ and is determined spectroscopically to be an A0 main sequence star. What is its distance? (See Sparke & Gallagher Table 1.4.)
Solution:
The flux(f) magnitude(m) relation is $m = -2.5\log(f)$. So the flux of each stars is given by.

$$f = 10^{-\frac{m}{2.5}}$$

The flux is additive so the total flux of binary is just twice of this $f_{tot} = 2 \times f = 2 \times 10^{-\frac{m}{2.5}}$. Now the apparent magnitude (m) of the binary is:

$$m = -2.5\log(f_{tot}) = -2.5\log\left(2 \times 10^{-\frac{m}{2.5}}\right) = -2.5\left(\log(2) - \frac{m}{2.5}\right) = m - 0.75$$

So the apparant magnitude of binary is $m - 0.75$.
Given that the apparant magnitude of the star is $m_V = 10$, As it is a A_0 from the table the value for absolute magnitude is found to be $M_V = 0.80$. We know that the relation between the absolute magnitude(M_V) and apparant magnitude(m_V) and the distane of the star (r),

$$\begin{aligned} M_V - m_v &= 5(1 - \log(r)) \qquad \text{where r is in parsec} \\ -9.2 &= 5(1 - \log(r)) \\ \log(r) &= 2.84 \\ r &= 10^{2.84} = 691.83pc \end{aligned}$$

So the distane of the star is 691.83pc $\square$

3. If the mass function for stars follows the Salpeter distribution, with

$$\xi(M)\frac{dN}{dM} = AM^{2.35}$$

(where dN is the number of stars with masses between M and M + dM; see Sparke & Gallagher, p. 66), for $M_l < M < M_u$, with Ml Mu, and the stellar massluminosity relation is

$$L(M) \propto M^4,$$

show that the total number and total mass of stars depend mainly on Ml , while the total luminosity depends mainly on Mu. Specifically, for Ml = $0.2M_\odot$ and Mu = $100M_\odot$, calculate the masses M1 and M2 such that 50% of the total mass is contained in stars with $M < M1$, while 50% of the total luminosity is contained in stars with $M > M2$.
Solution:
The total mass of dN stars is MdN. So the total mass of the range is

$$M_{tot} = \int_{M_l}^{M_u} MdN = \int_{M_l}^{M_u} M \cdot AM^{-2.35}dN = A\int_{M_l}^{M_u} M^{-1.35}dN = \frac{A}{-0.35}\left(M_u^{-0.35} - M_l^{-0.35}\right)$$

Since we have to find $M_u = M_1$ such that half the total mass between $0.2M_\odot$ and $100M_\odot$ is to be equal to the total mass in the range $0.2M_\odot$ and M_1. Let's suppose $M_1 = \alpha M_\odot$. So,

$$\begin{aligned} \frac{1}{2}\left[\frac{A}{-0.35}\left\{(100M_\odot)^{-0.35} - (0.2M_\odot)^{-0.35}\right\}\right] &= \left[\frac{A}{-0.35}\left\{(\alpha M_\odot)^{-0.35} - (0.2M_\odot)^{-0.35}\right\}\right] \\ \frac{1}{2}\left[100^{-0.35} - 0.2^{-0.35}\right] &= \left[\alpha^{-0.35} - 0.2^{-0.35}\right] \\ \alpha &= \left[\frac{100^{-0.35} + 0.2^{-0.35}}{2}\right]^{\frac{1}{-0.35}} \\ \alpha &= 1.06 \end{aligned}$$

So for the star in the range $0.2M_\odot$ to $1.06M_\odot$ have half the total number of the stars. The luminosity of each star of mass M is proportional to M^4 and there are dN such stars. So the total luminosity of starts between mass M and $M + dM$ is proportional to $M^4 dN$, So the total luminosity of the range M_l and M_u is a constant times

$$L_{tot} = \int_{M_l}^{M_u} M^4 dN = \int_{M_l}^{M_u} M^4 \cdot AM^{-2.35} dN = A \int_{M_l}^{M_u} M^{1.65} dN = \frac{A}{2.65}\left(M_u^{2.65} - M_l^{2.65}\right)$$

Since we have to find $M_l = M_2$ such that half the total luminosity between $0.2M_\odot$ and $100M_\odot$ is to be equal to the total luminosity in the range M_l and $100M_\odot$. Let's suppose $M_1 = \beta M_\odot$. So,

$$\frac{1}{2}\left[\frac{A}{2.65}\left\{(100M_\odot)^{2.65} - (0.2M_\odot)^{2.65}\right\}\right] = \left[\frac{A}{2.65}\left\{(100M_\odot)^{2.65} - (\beta M_\odot)^{2.65}\right\}\right]$$

$$\frac{1}{2}\left[100^{2.65} - 0.2^{2.65}\right] = \left[100^{2.65} - \beta^{2.65}\right]$$

$$\beta = \left[\frac{100^{2.65} + 0.2^{2.65}}{2}\right]^{\frac{1}{2.65}}$$

$$\beta = 76.98$$

$M_2 = 76.98M_\odot$ So the stars in the range $77M_\odot$ to $100M_\odot$ have half the luminosity as that of total stars in the range. □

4. Astronomers often approximate the stellar mass function (M) by a Salpeter power-law with a low-mass cutoff, but the Kroupa distribution

$$\xi(M) = \begin{cases} CM^{-0.3} & \text{for } M \le 0.1M_\odot \\ BM^{-1.3} & \text{for } 0.1M_\odot < M \le 0.5M_\odot \\ AM^{-2.35} & \text{for } M > 0.5M_\odot \end{cases}$$

is actually a much better description [A is the same as in part (a) and the other constants B and C are chosen to ensure that is continuous.] If the upper mass limit in all cases is Mu $= 100M_\odot$ and we assume the same simplified massluminosity relation as in part (a), what low-mass cutoff Ml must be chosen in order that the truncated power-law has the same (i) total number of stars, (ii) total mass, and (iii) total luminosity as the Kroupa distribution?

Solution:

Since the given function $\xi(M)$ should be contunuous, each piece should have equal value at the boundary.

$$B(0.5M_\odot)^{-1.3} = A(0.5M_\odot)^{-2.35} \Rightarrow B = 2.070M_\odot^{-1.05}A$$
$$C(0.1M_\odot)^{-0.3} = B(0.1M_\odot)^{-1.35} \Rightarrow C = 10M_\odot^{-1}B = 20.70M_\odot^{-2.05}A$$

The total number of stars given by Kroupa distribution is

$$N = \int_0^{100M_\odot} \xi(M)dM = \int_0^{0.1M_\odot} 20.70AM_\odot^{-2.05}M^{-0.3}dM + \int_{0.1M_\odot}^{0.5M_\odot} 2.07AM_\odot^{-1.05}M^{-1.3}dM + \int_{0.5M_\odot}^{100M_\odot} AM^{-2.35}dM$$

$$N = 5.90AM_\odot^{-1.35} + 5.27AM_\odot^{-1.35} + 1.88AM_\odot^{-1.35} = 13.05M_\odot^{-1.35}A$$

Also the total number of star given by salpeter distribution with lower mass limit as $(\alpha M_\odot)$

$$N = \int_{\alpha M_\odot}^{100M_\odot} AM^{-2.35}dM$$
$$= 0.74(\alpha M_\odot)^{-1.35}A - 0.0014M_\odot^{-1.35}A$$

Equating these values

$$13.05M_\odot^{-1.05}A = 0.74(\alpha M_\odot)^{-1.35}A - 0.0014M_\odot^{-1.35}A$$
$$\Rightarrow \alpha^{-1.35} = 17.63$$
$$\Rightarrow \alpha = 0.11$$

Therefore the lower limit is $0.11M_\odot$ if Salpeter distribution and Kroupa distribution have the same number of stars.

Working in the units of $M_\odot = 1$ and $A = 1$:
The total Mass of stars given by Kroupa distribution is

$$M = \int_0^{100} M\xi(M)dM = \int_0^{0.1} 20.70M^{0.7}dM + \int_{0.1}^{0.5} 2.07M^{-0.3}dM + \int_{0.5}^{100} M^{-1.35}dM$$
$$M = 0.24 + 1.23 + 3.07 = 4.54$$

Also the total Mass of star given by salpeter distribution with lower mass limit as $(\alpha M_\odot)$

$$M = \int_\alpha^{100} M \times M^{-2.35}dM$$
$$= 2.85\alpha^{-0.35} - 0.57$$

Equating these values

$$4.54 = 2.85\alpha^{-0.35} - 0.57$$
$$\Rightarrow \alpha^{-0.35} = 1.79$$
$$\Rightarrow \alpha = 0.19$$

Therefore the lower limit is $0.19M_\odot$ for the Salpeter distribution and Kroupa distribution to have the same total mass.

The total Luminosity of stars given by Kroupa distribution is

$$L = \int_0^{100} M^4\xi(M)dM = \int_0^{0.1} 20.70AM^{3.7}dM + \int_{0.1}^{0.5} 2.07M^{2.7}dM + \int_{0.5}^{100} M^{1.65}dM$$
$$L = 8.78 \times 10^{-5} + 0.042 + 75292.85 = 75292.89$$

Also the total Luminosity of all stars given by salpeter distribution with lower mass limit as $(\alpha M_\odot)$

$$L = \int_\alpha^{100} M^4 \times M^{-2.35}dM$$
$$= 75292.92 - 0.37\alpha^{2.65}$$

Equating these values

$$75292.89 = 75292.92 - 0.37\alpha^{2.65}$$
$$\Rightarrow \alpha^{2.65} = 0.07$$
$$\Rightarrow \alpha = 0.36$$

Therefore the lower limit is $0.36M_\odot$ for Salpeter distribution and Kroupa distribution have the same Luminosity.

$\square$

5. (a) Use Gausss law to derive an expression for the gravitational force in the z direction due to an infinite sheet of surface density Σ lying in the xy plane. (b) A star has velocity 30 km/s perpendicular to the Galactic plane as it crosses the plane, and is observed to have a maximum departure above the plane of 500 pc. Approximating the disk as an infinite gravitating sheet of matter, estimate its surface density Σ (i) in kgm^2 and (ii) in $M_\odot pc^{-2}$

Solution:

The gravitational flux(Φ) thourgh a closed surface enclosing mass M_{encl} is

$$\Phi = 4\pi G M_{encl} \tag{1}$$

If we assume the galactic plane as an infinite sheet of mass uniformly distributed over a surface with surface density Σ and we take the Gaussian surface as a cylynder of radius a perpendicular to the plane, then the total mass included within the cylinder would be $M_{encl} = \text{Area} \times \Sigma = \pi a^2 \Sigma$. But the total surface area of cylinder that is perpendicular(z direction) to the Plane is $2\pi a^2$. If E is the Gravitational field at the cylinder surface, then total flux (Φ) through the area is $E \times 2\pi a^2$ Substuting the values of Φ and M_{encl} in (1) we get.

$$2\pi a^2 E = 4\pi G(\pi a^2 \Sigma)$$
$$\Rightarrow E = 2\pi G \Sigma$$

So the gravitatational force per unit mass in the z direction is $2\pi G\Sigma$.

Given that a star with velocity $v = 30km/s$ and travels a max distance of $s = 500pc = 1.543 \times 10^{19}m$. Sine the gravitational field is constant and is independent of distance above the galactic plane. We can use the constant accleration kinematics relation $v_f^2 - v_i^2 = 2as$. Since the speed at maximuh distance is zero.

$$a = \frac{v_i^2}{2s}$$

But the accleration $a = 2\pi G\Sigma$

$$\Sigma = \frac{v_i^2}{4\pi G s} = \frac{(3 \times 10^4)^2}{2 \times 1.543 \times 10^{19} \times 4\pi \times 6.672 \times 10^{-11}} = 0.069 kgm^{-2}$$

Since $1kg = 5.02 \times 10^{-31} M_\odot$ and $1m^{-2} = 9.52 \times 10^{32} pc^{-2}$

$$\Sigma = 0.069 \times 5.02 \times 10^{-31} \times 9.52 \times 10^{32} M_\odot pc^{-2} = 33.30 M_\odot pc^{-2}$$

So the surface mass density Σ for the given planar galaxy is $0.069kgm^2 \equiv 33.30 M_\odot pc^{-2}$. □