PHYS 522: Statistical Mechanics

Homework #4

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1. Derive the virial expansion of the ideal Bose gas by inverting the relation $n\lambda^3 = g_{3/2}(z)$ series to express z in terms of $n\lambda cb$ and the substitute it in the P/KT equation. Using this expression derive the expansion for C_v/Nk valid at high temperature limit.
Solution:
For high temperature $N_0 \ll N$ the relation can be written as

$$n\lambda^3 = g_{3/2}(z) = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \dots$$

To invert the series with usual technique we write the z as a power series in $n\lambda^3$ as

$$z = c_1(n\lambda^3) + c_2(n\lambda^3)^2 + c_3(n\lambda^3)^3 + \dots$$

substuting the value of z into the first series we get

$$n\lambda^3 = \left[c_1(n\lambda^3) + \dots\right] + \left[\frac{\left(c_1(n\lambda^3) + c_2(n\lambda^3)^2\right)^2}{2^{3/2}}\right] + \dots$$

Comparing the coefficients of like powers of λ^3 in both sides we get

$$c_1 = 1; \qquad c_2 + \frac{c_1^2}{2^{-3/2}} = 0 \qquad c_3 + \frac{2c_1c_2}{2^{-3/2}} + \frac{c_1^3}{3^{-3/2}} = 0$$

Writing similarly we get

$$c_1 = 1; c_2 = \frac{-1}{2^{-3/2}} \qquad c_3 = \frac{1}{4} - \frac{1}{3^{-3/2}}$$

Now the expression $\ln \mathcal{Q}$ becomes

$$\frac{PV}{NkT} = \frac{1}{n\lambda^3}\left(z + \frac{z^2}{2^{-5/2}} + \frac{z^3}{3^{-5/2}} + \dots\right)$$

Substuting the value of z from the series in $n\lambda^3$ with the various coefficeints $c_1, c_2 \dots$ we get

$$\frac{PV}{NkT} = \sum_{l=1}^{\infty} a_l \left(\frac{\lambda^3}{v}\right)^{l-1}$$

This is the required virial expansion of the expression. Now for the specific heat at constant volume we have to find out $\frac{\partial U}{\partial T}$, this can be simplified as

$$\frac{C_v}{Nk} = \frac{1}{Nk}\left(\frac{\partial U}{\partial T}\right)_{N,V} = \frac{3}{2}\left[\frac{\partial}{\partial T}\left(\frac{PV}{Nk}\right)\right]_v$$

In similar fashion for the expansion of $g_{5/2}(z)$ we get

$$\frac{C_v}{Nk} = \sum_{l=1}^{\infty} \frac{3}{2}\frac{5-3l}{2} a_l \left(\frac{\lambda^3}{v}\right)^{l-1}$$

Substuting all the coefficient we get

$$\frac{C_v}{Nk} = \frac{3}{2}\left[1 + c_1\left(\frac{\lambda^3}{v}\right) + c_2\left(\frac{\lambda^3}{v}\right)^2 + \dots\right]$$

where teh coefficients are $c1 = 0.088, c_2 = 0.0065, \dots$ This is the expression of specific heat of bose gas correct at high temperature. □

2. **(Pathria & Beale, 7.3)** Combining equation 7.1.24 and 7.1.26, and making use of the first wo terms of formula (D.9) in Appendix D, show that, as T approaches T, from above the parameter $\alpha(=\ln z)$ of the ideal bose gas assumes the form

$$\alpha = \frac{1}{\pi}\left(\frac{3\zeta(3/2)}{4}\right)^2\left(\frac{T-T_c}{T}\right)^2$$

Solution:
We have from previous problem $n\lambda^3 = g_{3/2}(z)$. But at $\lambda = \lambda_c$ we have $z = 1$. But for $z = 1$

$$g_{3/2}(1) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \ldots = \zeta(3/2)$$

Substuting this in the expression for the critical temperature and taking the ratio

$$\frac{T}{T_c} \equiv \left(\frac{\lambda}{\lambda_c}\right)^2 = \left(\frac{g_{3/2}(z)}{\zeta(3/2)}\right)^{-\frac{2}{3}}$$

The expression for $g_{3/2}(z)$ can be expanded in termf os series the series expansion from appendix D.9 can be used to obtain

$$\frac{T}{T_c} = \left(\frac{\zeta 3/2 - 2\sqrt{\pi\alpha} + \ldots}{\zeta(3/2)}\right)^{-\frac{2}{3}}$$

Since we have $\alpha \ll 1$ we can make use of binomial expansion of the series

$$(1+x)^n \approx 1 + nx; \qquad x \ll 1$$

Using just the first two terms we get

$$\frac{T}{T_c} \approx 1 + 4\frac{\sqrt{\pi\alpha}}{3\zeta(3/2)}$$

Now, this expression can be simplified further to get

$$4\sqrt{\pi\alpha} = 3\zeta(3/2)\left(\frac{T-T_c}{T_c}\right)$$

Squaring both sides leds to

$$\alpha = \frac{1}{\pi}\left(\frac{3\zeta(3/2)(T-T_c)}{4T}\right)^2$$

This is the required expression. □

3. Derive in detailed steps the following expression for an ideal Bose gas.

$$\frac{C_v}{Nk} = \frac{15 g_{5/2}(z)}{4 g_{3/2}(z)} - \frac{9 g_{3/2}(z)}{4 g_{3/2}(z)}$$

Solution:
For ideal bose gas from 7.1.7 and 7.1.8 we get

$$\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z)$$

$$\frac{N-N_0}{V} = \frac{1}{\lambda^3} g_{3/2}(z)$$

At high temperature we can assume that $z \ll 1$ is very small and we can safely ignore N_0. We can take the ratio of these two quantities to get

$$\frac{PV}{NkT} = \frac{g_{5/2}(z)}{g_{3/2}(z)}$$

Also the internal energy can be calculated as

$$U \equiv -\left(\frac{\partial}{\partial \beta}\ln \mathcal{Q}\right)_{z,V} = kT^2\left(\frac{\partial}{\partial T}\left(\frac{PV}{KT}\right)\right)_{z,v} = \frac{3}{2}kT\frac{V}{\lambda^3}g_{5/2}(z)$$

Now the expression for the specific heat is

$$C_v = \frac{\partial U}{\partial T} = \left[\frac{\partial}{\partial T}\left(\frac{3}{2}T\frac{g_{5/2}(z)}{g_{3/2}(z)}\right)\right]_v$$

Now we can use the recurrance relation for the function $g(z)$ as

$$z\frac{\partial}{\partial z}g_\nu(z) = g_{\nu-1}(z)$$

Also since the function $g_{3/2}(z)$ is proportional to cube root of the square of the temperature we get

$$\left[\frac{\partial}{\partial T}g_{3/2}(z)\right]_v = -\frac{3}{2T}g_{3/2}(z)$$

Combining these two expressions we get

$$\frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_v = -\frac{3}{2T}\frac{g_{3/2}(z)}{g_{1/2}(z)}$$

Now carrying out the differentiation of the expression C_v we get

$$C_v = Nk\frac{3}{2}\frac{g_{5/2}(z)}{g_{3/2}(z)} + Nk\frac{\partial}{\partial T}\left(\frac{g_{5/2}(z)}{g_{3/2}(z)}\right)\frac{\partial z}{\partial T}$$

Using the previous expression for $\frac{\partial z}{\partial T}$ and using the product rule in the differentiation we get

$$\frac{C_v}{Nk} = \frac{3}{2}\left[\frac{5}{2}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{3}{2}\frac{g_{3/2}(z)}{g_{1/2}(z)}\right]$$

Simplifying the expression gives

$$\frac{C_v}{Nk} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}$$

This is the requred expression for the specific heat of bosons in high temperature limit. □

4. Prove the following for and atomic Bose gas with spin S
   (a) Its density of state is given by:

$$g(E) = 2\pi V(2S+1)\left(\frac{2m}{h^2}\right)^{3/2}E^{1/2}$$

Solution:
If we consider atomic non-interating atomic gas with spin S, then for each momentum state, there are $2S+1$ spin states. Then the grand partition function becomes

$$\mathcal{Q} = \prod_i \mathcal{Q}_i^{2S+1}$$

The grand potential becomes

$$\Phi = -kT\ln\mathcal{Q} = kT(2S+1)\sum_i \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right)$$

Approximating the sum with the integration we get

$$\Phi = kT(2S+1)\int_0^\infty \ln\Big(1+e^{-\beta(\varepsilon-mu)}\Big)g(E)dE$$

Here $g(E)$ is the density of states which cn be simplified for uniformaly distributed particles as

$$g(k)dk = \frac{4\pi k^2(2S+1)dk}{(2\pi/L)^3} = \frac{(2S+1)VK^2dk}{2\pi^2}$$

With volume $V = L^3$ and $E = \frac{\hbar^2 k^2}{2m}$ we get

$$g(E)dE = \frac{(2S+1)V\sqrt{E}dE}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$$

Using $\hbar = \frac{h}{2\pi}$ and writing the density of states we get

$$g(E) = 2\pi V(2S+1)\sqrt{E}\left(\frac{2m}{h}\right)^{3/2}$$

Which is the required density of states. □

(b) Then show that its Bose-Enistein temperature is given by

$$T_c = \frac{h^2}{2\pi mk}\left[\frac{n}{2.612(2S+1)}\right]^{2/3}$$

Solution:
Now the total number o particles N can be obtained as

$$N = \int_0^\infty \frac{g(E)dE}{e^{\beta(\varepsilon-\mu)}+1}$$

Substuting the density function we get

$$N = \left[2\pi V\left(\frac{2m}{h^2}\right)^{3/2}\right]\int_0^\infty \frac{\sqrt{E}dE}{z^{-1}e^{\beta E}-1}$$

The integrand can be recocnized as the einstein function $g_{3/2}(z)$. so we get

$$N = \frac{(2S+1)V}{\lambda^3}g_{3/2}(z)$$

For $T = T_C$ we can consider z comparable to unity, thus, we have $z = 1$, substuting this we get $g_{3/2}(1) = \zeta(3/2) = 2.612$

$$\frac{n\lambda^3}{2S+1} = \zeta(3/2) = 2.612$$

Making this substution and recocnizing $\lambda = \left[\frac{h^s}{2\pi mkT}\right]^{3/2}$ Rearrainging we get

$$T_c = \frac{h^2 2}{\pi mk}\left[\frac{n}{2.612(2S+1)}\right]$$

where we have made use of $n = \frac{N}{V}$. This gives the expression for the critical temperature of Bose gas. □