PHYS 517: Quantum Mechanics II

Homework #6

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May 31, 2018

1. **(Sakurai 5.1)** A simple harmonic oscillator (in one dimension) is subjected to perturbation

$$\lambda H_1 = bx,$$

where b is a real constant. You may assume without proof that

$$\langle u_{n'}|x|u_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1}\right)$$

(a) Calculate the energy shift of the ground state to lowest nonvanishing order.
Solution:
The the energy shift in unpertrubed r^{th} state ket $|n^r\rangle$ is given by

$$\Delta_n = b\left\langle n^0\middle|V\middle|n^z0\right\rangle + b^2\sum_{r\neq n}^{\infty}\frac{\left|\left\langle n^0\middle|V\middle|n^r\right\rangle\right|^2}{E_n^0 - E_r^0} + \dots$$

The first order correction for grond state term is

$$b\left\langle n^0\middle|V\middle|n^0\right\rangle = b\langle 0|x|0\rangle = b\sqrt{\frac{\hbar}{2m\omega}}\left(\delta_{0,1} + \delta_{0,1}\right) = 0$$

Matrix element $\langle n|x|n'\rangle$ clearly vanishes for when n and n' differ by any other quantity than one, 1. The infinite sum gives only single non vanishing term when $|n^r\rangle = |1\rangle$ in that case

$$b^2\sum_{r\neq n}^{\infty}\frac{\left|\left\langle n^0\middle|V\middle|n^r\right\rangle\right|^2}{E_n^0 - E_r^0} = b^2\frac{|\langle 0|x|1\rangle|^2}{\frac{1}{2}\hbar\omega - \frac{3}{2}\hbar\omega} = b^2\frac{\left|\sqrt{\frac{\hbar}{2m\omega}}\right|^2}{-\hbar\omega} = -\frac{b^2}{2m\omega^2}$$

This is the first nonvanishing term in the energy correction. □

(b) Solve this problem exactly and compate with your result in (1a).
Solution:
The total hamiltonian after pertubation becomes

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + bx$$

We can rearrange the terms by completing square for the last two terms to get

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x + \frac{b}{m\omega^2}\right)^2 - \frac{b^2}{2m\omega^2}$$

This is again a Hamiltonian for a simple harmonic oscillator. The added quantity with x doesnot change the behaviour of the oscillator as it is just coordinate translation, the last term is a constant energy term. This will give energy eigenvalues exactly as we had before with with an added constant of

$$-\frac{b^2}{2m\omega^2}$$

This is exactly what was found in (1a). □

2. **(Sakurai 5.2)** In nondegenerate time-independent perturbation theory, what is the probability of finding in a pertrubed energy eigenstate ($|k\rangle$) the corresponding unpertrubed eigenstate ($|k^{(0)}\rangle$)? Solve this upto terms of order g^2
Solution:
The probability of pertrubed state to be found in the unpertrubed state can be calculated by the normalization constant that normalizes the unpertrubed state ket. Let us suppose that Z_n normalizes the pertrubed state ket

$$|n\rangle_N = \sqrt{Z_n}\,|n\rangle$$

where $|n\rangle$ is the pertrubed state ket and $|n\rangle_N$ is the normalized ket. This can be found by the raltion

$${}_N\langle n|n\rangle_N = Z_n \langle n|n\rangle = 1$$

Using the expansion of pertrubed kets we get

$$\begin{aligned}
\Rightarrow \frac{1}{Z_n} = \langle n|n\rangle &= \left(\langle n^0| + \lambda\langle n^1| + \lambda^2\langle n^2| + \ldots\right) \times \left(|n^0\rangle + \lambda|n^1\rangle + \lambda^2|n^2\rangle + \ldots\right) \\
&= 1 + \lambda^2\langle n^1|n^1\rangle + \mathcal{O}(\lambda^3) \\
&= 1 + \lambda^2 \sum_{k\neq n} \frac{|\langle k|V|n\rangle|^2}{(E_n^0 - E_k^0)^2} + \mathcal{O}(\lambda^3)
\end{aligned}$$

So the probability is given by Z_n which is jus the inverse of above quantity using the binomial expansion we get

$$Z_n = \left[1 + \lambda^2 \sum_{k\neq n} \frac{|\langle k|V|n\rangle|^2}{(E_n^0 - E_k^0)^2} + \mathcal{O}(\lambda^3)\right]^{-1} = 1 - \lambda^2 \sum_{k\neq n} \frac{|\langle k|V|n\rangle|^2}{(E_n^0 - E_k^0)^2} + \mathcal{O}(\lambda^3)$$

This expression gives the probability of finding the pertrubed ket in the original unpertrubed state ket. □

3. Consider a one dimensional infinite square well potential

$$H_0 = \frac{p^2}{2m} + U(x)$$

where

$$U(x) = \begin{cases} 0 & \text{if } 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

Now consider a perturbation by placing a delta function at the center of the well

$$V(x) = \frac{\hbar^2}{mL}\delta\left(x - \frac{L}{2}\right)$$

(a) What is the unpertrubed energy levels and wavefunctions?
Solution:
The schrodingers equation is

$$H|\psi\rangle = E|\psi\rangle$$

Let the wave function of the system be $\psi(x) = \langle x|\psi\rangle$. Also the hamiltonian can be written with operator p as $p^2 = (i\hbar\nabla)^2 \equiv -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ so the schrodingers equation becomes

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_n(x) = E_n\psi_n(x)$$

This is a well known seond order differential equation whose solution are well knows to be

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) \qquad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

These are the required wavefunction and energy levels for th unpertrubed system. □

(b) What is the first order correction to the energy shift?
Solution:
The first order correction in energy shift is given by

$$\begin{aligned}\langle\psi_1|V|\psi_1\rangle &= \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right)\delta\left(x-\frac{L}{2}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right)dx \\ &= \frac{2}{L}\sin^2\left(\frac{\pi}{2}\right) = \frac{2}{L}\end{aligned}$$

So the first order energy correction is ${}^2\!/_L$. □

(c) What is the first order correction to the ground state wavefunction?
Solution:
The correction for wave function is given by the expression

$$|n\rangle = \left|n^0\right\rangle + \lambda\sum_{k\neq n}\left|k^0\right\rangle\frac{\langle n|V|k\rangle}{E_n^0 - E_k^0} + \mathcal{O}(\lambda^2)$$

For ground state $n = 1$ so the correction becomes

$$\langle 1|V|k\rangle = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right)\delta\left(x-\frac{L}{2}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right)dx = \frac{2}{L}\sin\left(\frac{k\pi}{2}\right)$$

The Energy difference with the ground state are

$$E_1 - E_k = \frac{\pi^2\hbar^2}{2mL^2}\left(1^2 - k^2\right)$$

So the pertrubed ground state wave function becomes

$$|1\rangle = \psi_1(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right) + \sum_{k\neq 1}^{\infty}\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right)\left(\frac{4Lm}{\pi^2\hbar^2}\right)\frac{\sin\left(\frac{k\pi}{2}\right)}{1-k^2}$$

This is the required pertrubed eigenstate for the gorund state of the potential. □

(d) What is the second order correctio to the energy shift specially for the ground state?
Solution:
The second order shift in energy is given by

$$\sum_{r\neq n}^{\infty}\frac{\left|\left\langle n^0\middle|V\middle|n^r\right\rangle\right|^2}{E_n^0 - E_r^0}$$

For the state waefunctions we have

$$\langle n|V|r\rangle = \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) \delta\left(x - \frac{L}{2}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{r\pi}{L}x\right) dx = \frac{2}{L} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{r\pi}{2}\right)$$

The Energy differences are

$$E_n - E_r = \frac{\pi^2\hbar^2}{2mL^2}\left(n^2 - r^2\right)$$

Substuting this back we get

$$\Delta_n = \frac{4Lm}{\pi^2\hbar^2} \sum_{n\neq r}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{r\pi}{2}\right)}{n^2 - r^2}$$

For ground state $n = 1$ so the energy correction for ground state becomes

$$\Delta_1 = \frac{4Lm}{\pi^2\hbar^2} \sum_{r\neq 1}^{\infty} \frac{\sin\left(\frac{r\pi}{2}\right)}{1 - r^2}$$

These are the required energy corrections. □

4. Calculate the ground state wave function for Simple Harmonic Oscillator with the pertubation $V = \frac{1}{2}\varepsilon m\omega^2 x^2$.

Solution:

The ground state wave function without the perturbation for simple harmonic oscillator is

$$\langle x|0\rangle = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega x^2}{2\hbar}\right]$$

The perturbation scales the frequency by a factor of $\sqrt{1+\varepsilon}$ sustuting this in the frequency of unpertrubed state we get

$$\langle x|0\rangle = \sqrt[4]{\frac{m\sqrt{(1+\varepsilon)}\omega}{\pi\hbar}} \exp\left[-\frac{m\sqrt{(1+\varepsilon)}\omega x^2}{2\hbar}\right]$$

Since the perturbation factor $\varepsilon \ll 1$ we can use the binomial expansion as

$$(1+\varepsilon)^{1/8} \approx 1 + \frac{\varepsilon}{8} \qquad (1+\varepsilon)^{1/2} \approx 1 + \frac{\varepsilon}{2}$$

Using these we get

$$\begin{aligned}\langle x|0\rangle &= \sqrt[4]{\frac{m\omega}{\pi\hbar}} \left(1 + \frac{\varepsilon}{8}\right) \exp\left[-\frac{m\omega x^2}{2\hbar}\left(1 + \frac{\varepsilon}{2}\right)\right] \\ &\approx \sqrt[4]{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega x^2}{2\hbar}\right] + \varepsilon\left[\frac{1}{8} - \frac{m\omega x^2}{4\hbar}\right] \sqrt[4]{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega x^2}{2\hbar}\right]\end{aligned}$$

This is the required pertrubed energy eigenstate. □