PHYS 517: Quantum Mechanics II

Homework #5

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1. For two spin $\frac{1}{2}$ particles, ingornign orbital angular momentu , the singlet state is

$$|s=0;m=0\rangle = \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle)$$

Verify by explicitly rotatin the state about the y-axis by angle 0 so that it is rotationally invariant
Solution:
Writing the operator $S_y = \frac{1}{2i}(S_+ - S_-)$. The rotated state is

$$\begin{aligned}
&e^{-\frac{iS_y}{\hbar}}\frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[\exp\left(-\frac{\theta}{2\hbar}\sqrt{\left(\frac{1}{2}-\frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}+1\right)}\right)|+-\rangle - \exp\left(-\frac{\theta}{2\hbar}\sqrt{\left(-\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}+1\right)}\right)|-+\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[e^0|+-\rangle - e^0|-+\rangle\right] \\
&= \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle)
\end{aligned}$$

So the state is really lotationally invariant under rotation on θ rotation on $y-axis$ □

2. A system consists of three independent sybsystems with angular moementum J_1, J_2 and J_3 respectively such that $[J_{ai}, J_{bj}] = i\hbar\varepsilon_{ijk}J_{ak}\delta_{ab}$ where the subsystems indecies a and b are $1, 2$ or 3.

   (a) We know that the simultaneous eigenket for the six operators $\left\{\mathbf{J_2}^2, \mathbf{J_2}^2, \mathbf{J_3}^2, J_{1z}, J_{2z}, J_{3z}\right\}$ is a choice for base ket. Now construct another coice for a base ket to describe the system which includes the operator $\mathbf{J} = \mathbf{J_1} + \mathbf{J_2} + \mathbf{J_3}$. Briefly explain or show how the six operators commute with each other making it a valid base ket.
   Solution:
   Defining $\mathbf{J} = \mathbf{J_1} + \mathbf{J_2} + \mathbf{J_3}$, I think $\left\{\mathbf{J}^2, J_{12} = \mathbf{J_1}^2 + \mathbf{J_2}^2, J_{23} = \mathbf{J_2}^2 + \mathbf{J_3}^2, J_{31} = \mathbf{J_3}^2 + \mathbf{J_1}^2, J_z\right\}$ works as six operators. Since $\left[\mathbf{J}^2, \mathbf{J}_i^2\right] = 0$, this implies $\left[\mathbf{J}^2, J_{ij}\right] = 0, i, j \in 1, 2, 3$.
   Also since the given simultaneous operators $\mathbf{J}_i^2$ commute with each of J_{jz} this implies that

$$\begin{aligned}
\left[\mathbf{J}^2, J_{ij}\right] &= \left[\mathbf{J}^2, J_i^2 + J_j^2\right] = \left[\mathbf{J}^2, J_i^2\right] + \left[\mathbf{J}^2, J_j^2\right] = 0 + 0 = 0 \\
\left[J_{ij}, J_{kz}\right] &= \left[J_i^2 + J_j^2, J_{kz}\right] = \left[J_i^2, J_{kz}\right] + \left[J_j^2, J_{kz}\right] = 0 + 0 = 0 \\
\left[\mathbf{J}^2, J_{kz}\right] &= \left[\mathbf{J_1^2} + \mathbf{J_2^2} + \mathbf{J_3^2} + \mathbf{2J_1J_2} + \mathbf{2J_2J_3} + \mathbf{2J_3J_1}, J_{kz}\right] = 0
\end{aligned}$$

   Since each of these operators are independent of each other, these form a complete set of commuting operators. □

   (b) Write the state $|j_1 = 1, j_2 = 1, j_3 = 1, j_{1z} = 1, j_{2z} = 1, j_{3z} = 1\rangle$ in the other representation. Then use the ladder operator to find the Clebsch-Gordon coefficients for th enext lowered states.
   Solution:

The maximum value of j in the system with eivenvalue of $\mathbf{J} = \mathbf{J_1} + \mathbf{J_2} + \mathbf{J_3}$ is $j = m_1 + m_2 + m_3 = 3$. We can denote the system with these eivenvaues of $\mathbf{J}^2$ operator and J_z operator as

$$|j=3; m=3\rangle = |j_1 = 1, j_2 = 1, j_3 = 1, j_{1z} = 1, j_{2z} = 1, j_{3z} = 1\rangle$$

Dropping the eigenvalues of J_i operators and writing just the eigenvalues of J_{kz}, as there is no ambiguity Operating by J_- operator on both sides we get

$$\begin{aligned}
J_- |j=3; m=3\rangle &= (J_{1-} + J_{2-} + J_{3-}) |j_{1z} = 1, j_{2z} = 1, j_{3z} = 1\rangle \\
\sqrt{(3+3)(3-3+1)}\hbar\, |j=3; m=2\rangle &= \sqrt{(1+1)(1-1+1)}\hbar\, |011\rangle + \sqrt{(1+1)(1-1+1)}\hbar\, |101\rangle \\
&= \quad + \sqrt{(1+1)(1-1+1)}\hbar\, |110\rangle \\
\sqrt{6}\, |j=3; m=2\rangle &= \sqrt{2}\, |011\rangle + \sqrt{2}\, |101\rangle + \sqrt{2}\, |110\rangle \\
|j=3; m=2\rangle &= \frac{1}{\sqrt{3}} \Big[\ |011\rangle + |101\rangle + |110\rangle \Big]
\end{aligned}$$

So the Clebsch-Gordon coefficients for this system are $\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$ □

3. **(Sakurai 3.24)** We are to add angular momenta $j_1 = 1$ and $j_2 = 1$ to form $j = 2, 1$ and 0 states. Using either the ladder operator method or the recursion relation, express all $\{j, m\}$ eigenkets in terms of $|j_1, j_2; m_1, m_2\rangle$, Write your answer as

$$|j=1, m=1\rangle = \frac{1}{\sqrt{2}} |+, 0\rangle - \frac{1}{2} |0, +\rangle, \ldots,$$

where + and 0 stand for $m_{1,2} = 1, 0$, respectively.

Solution:

Since maximum value of $j = j_1 + j_2 = 2$. The state corresponding to this added momenta must be $|22\rangle = |m_1, m_2\rangle$. Since we denote $|m_1 = 1; m_2 = 1\rangle = |{+}{+}\rangle$ and $|m_1 = 1; m_2 = 0\rangle = |{+}0\rangle$ and $|m_1 = 1; m_2 = -1\rangle = |{+}{-}\rangle$we obtain

$$|22\rangle = |{+}{+}\rangle$$

Applying lowering operator on both sides gives

$$L_- |22\rangle = (L_{1-} + L_{2-}) |{+}{+}\rangle \Rightarrow \sqrt{4 \cdot 1}\hbar\, |21\rangle = \sqrt{2}\hbar\, |0{+}\rangle + \sqrt{2}\, |{+}0\rangle \quad \Rightarrow |21\rangle = \frac{1}{\sqrt{2}} \Big[\ |0{+}\rangle + |{+}0\rangle \Big]$$

Similarly applyig lowering operator again gives

$$\sqrt{3 \cdot 2}\, |20\rangle = \frac{1}{\sqrt{2}} \Big[\sqrt{2}\, |{-}{+}\rangle + \sqrt{2}\, |00\rangle + \sqrt{2}\, |00\rangle + \sqrt{2}\, |{+}{-}\rangle\Big] \Rightarrow |20\rangle = \frac{1}{\sqrt{6}} \Big[\ |{-}{+}\rangle + 2\, |00\rangle + |{+}{-}\rangle \Big]$$

Since $L_{1-} |{-}{\pm}\rangle = 0$ and $L_{2-} |{\pm}{-}\rangle = 0$owering again gives

$$\sqrt{6}\, |2; -1\rangle = \frac{1}{\sqrt{6}} \Big[\sqrt{2}\, |{-}0\rangle + 2\sqrt{2}\, |{-}0\rangle + 2\sqrt{2}\, |0{-}\rangle + \sqrt{2}\, |0{-}\rangle\Big] \Rightarrow |2-1\rangle = \frac{1}{\sqrt{2}} [|{-}0\rangle + |0{-}\rangle]$$

Lowering one more time gies

$$\sqrt{4}\, |2, -2\rangle = \frac{1}{\sqrt{2}} \Big[\sqrt{2}\, |{-}{-}\rangle + \sqrt{2}\, |{-}{-}\rangle\Big] \Rightarrow |2, -2\rangle = |{-}{-}\rangle$$

The state can't be further lowered since all lowered states from now will be null kets. □