PHYS 517: Quantum Mechanics II

Homework #4

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1. **(Sakurai 3.18)** A particle in a spherically symmetrical potential is known to be in an eigenstat of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively. Prove that the expectation values between $|lm\rangle$ states satisfy

$$\langle L_x \rangle = \langle L_y \rangle = 0, \qquad \left\langle L_x^2 \right\rangle = \left\langle L_y^2 \right\rangle = \frac{\left[l(l+1)\hbar^2 - m^2\hbar^2\right]}{2}$$

Intrepret the result semiclassically.
Solution:
Since the definition of the operators $L_\pm = L_x \pm L_y$ these relations can be rearranged in to the oprators the operators

$$L_x = \frac{L_+ + L_-}{2} \qquad L_y = \frac{L_+ - L_-}{2i}$$

The expectation value of operator L_x is

$$\begin{aligned}
\langle L_x \rangle = \langle lm|L_x|lm\rangle &= \langle lm|\frac{L_+ + L_-}{2}|lm\rangle \\
&= \frac{1}{2}\langle lm|L_+|lm\rangle + \frac{1}{2}\langle lm|L_-|lm\rangle \\
&= \frac{1}{2}\langle lm|C_+|lm+1\rangle + \frac{1}{2}\langle lm|C_-|lm+1\rangle \\
&= 0 + 0 = 0
\end{aligned}$$

Similarly for L_y the expectation value is zero. The L_x^2 opeartor can be expanded into

$$\begin{aligned}
L_x^2 &= \left[\frac{L_+ + L_-}{2}\right]\left[\frac{L_+ + L_-}{2}\right] \\
&= \frac{1}{4}\left(L_+^2 + L_+L_- + L_-L_+ + L_-^2\right)
\end{aligned}$$

But the expectattion value of L_+^2 and L_-^2 are both zero because they raise and lower the state ket twice which are othognoal to each other.
Now the expectation value reduces to

$$\left\langle L_x^2 \right\rangle = \frac{1}{4}\langle L_+L_- + L_-L_+\rangle$$

But

$$L_+L_- + L_-L_+ = L_x^2 - iL_xL_y + iL_yL_x + L_y^2 + L_x^2 + iL_xL_y - iL_yL_x + L_y^2 = 2(L_x^2 + L_x^2) = 2(\mathbf{L}^2 - L_z^2)$$

Using this to find the expectation value of L_x^2 we get

$$\left\langle L_x^2 \right\rangle = \frac{1}{4}\langle L_+L_- + L_-L_+\rangle = \frac{1}{2}\left\langle \mathbf{L}^2 - L_2^2 \right\rangle = \frac{1}{2}\left(\hbar^2 l(l+1) + \hbar^2 m^2\right)$$

Similarly the expectatin value of L_y^2 is same as for L_x^2 and they are arequal. □

2. **(Sakurai 3.19)** Suppose a half-integer $l$value, say $\frac{1}{2}$, were allowed for orbital angular momentum. From

$$L_+ Y_{1/2,1/2}(\theta,\phi) = 0$$

we may deduce, as usual

$$Y_{1/2,1/2}(\theta,\phi) \propto e^{i\phi/2}\sqrt{\sin\theta}$$

Now try to construct $Y_{1/2,-1/2}(\theta,\phi)$ by (a) applying L_- to $Y_{1/2,1/2(\theta,\phi)}$; and (b) using $L_- Y_{1/2,-1/2}(\theta,\phi) = 0$. Show that the two procedures lead to contradictory result.
Solution:
Applying L_- on the given state $Y_{1/2,1/2}$ we get

$$\begin{aligned} Y_{1/2,-1/2}(\theta,\phi) &= -i\hbar e^{-i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\right) e^{i\phi/2}\sqrt{\sin\theta} \\ &= i\hbar e^{-i\phi}(-1)e^{-i\phi/2}\frac{1}{2}\frac{\cos\theta}{\sqrt{\sin\theta}} + i\hbar\cot\theta\frac{i}{2}e^{i\phi/2}\sqrt{\sin\theta} \\ &= -\hbar e^{-i\phi/2}\frac{\cos\theta}{\sqrt{\sin\theta}} \end{aligned}$$

checking to see if $L_- Y_{1/2,-1/2}(\theta,\phi) = 0$

$$\begin{aligned} L_- Y_{1/2,-1/2}(\theta,\phi) &= -i\hbar e^{i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right) e^{-i\phi/2}\frac{\cos\theta}{\sqrt{\sin\theta}}(-\hbar) \\ &= i\hbar^2 e^{-i\phi}\left(-i\left(-\frac{\sin\theta}{\sqrt{\sin\theta}} - \frac{1}{2}\frac{\cos^2\theta}{\sqrt{\sin^3\theta}}\right)e^{-i\phi/2} - \cot\theta\left(-i\frac{1}{2}\right)e^{-i\phi/2}\sqrt{\sin\theta}\right) \\ &= \hbar^2 e^{-3i\phi/2}\left(\frac{1}{\sqrt{\sin^3\theta}}\left[-2\sin^2\theta - \cos^2\theta + \frac{1}{2}\sin 2\theta\right]\right) \end{aligned}$$

The last expression is not zero which contradicts our proposition that tere exists a half integer $l-$value. □

3. **(Sakurai 3.20)** Consider an orbital angular-moemntum eigenstate $|l=2, m=0\rangle$. Suppose this sate is rotated by anangle β about $y-$axis. Find the probability for the new state to be found in $m = 0, \pm 1$ and ± 2. (The spherical harmonics for $l = 0, 1$ and 2 may be useful).
Solution:
Let the arbitrary state be $|P\rangle = |l=2; m=0\rangle$ the state kaet in the rotated system is $|P\rangle_R = \mathcal{D}(0,\beta,0)\,|P\rangle$ This reotaed state can be calculated as

$$\begin{aligned} \mathcal{D}_R(0,\beta,0)\,|P\rangle &= \sum_{m'} |l=2; m'\rangle\langle l=2; m'|\,\mathcal{D}_R(0,\beta,0)\,|l=2, m=0\rangle \\ &= \sum_{m'} |l=2; m'\rangle\,\mathcal{D}^{(2)}_{m',0}(0,\beta,0) = \sum_{m'} |l=2; m'\rangle\sqrt{\frac{4\pi}{5}}Y_2^{m'}(\beta,0)^* \end{aligned}$$

Thus the probability of finding the rotated state same as the original stae is

$$|\langle P|\mathcal{D}_R|P\rangle|^2 = \left|\sum_{m'}\langle l=2, m=0|l=2; m'\rangle\sqrt{\frac{4\pi}{5}}Y_2^{m'}(\beta,0)^*\right|^2 = \left|\sqrt{\frac{4\pi}{5}}Y_2^{m}(\beta,0)^*\right|^2$$

This is the required probability of finding the rotated state in original state.

Now for $m = 0$ we have $Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\beta - 1)$ this gives the probability $\frac{1}{4}(3\cos^2\beta - 1)^2$.

For $m = \pm 1$ we have $Y_{2,\pm 1} = \sqrt{\frac{15}{8\pi}}(\sin\beta\cos\beta)$ this gives the probability $\frac{3}{4}\sin^2\beta\cos^2\beta$.

For $m = \pm 2$ we have $Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}(\sin^2\beta)$ this gives the probability $\frac{3}{8}\sin^4\beta$. □