# PHYS 517: Quantum Mechanics II

Homework #3

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1. Expand the matrix

$$\mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma) = e^{-i(m'\alpha+m\gamma)}\left\langle j,m'\right|\exp\left(\frac{-iJ_y\beta}{\hbar}\right)\left|j,m\right\rangle .$$

**Solution:**
Clearly the order of matrix depends upon the value of $j$. The range of values for $m$ are constrained by the value of $j$ $j=1$, the matrix becomes

$$d^{(1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

□

2. **(Sakurai 3.13)** An angular-momentum eigenstate $|j, m = m_{max} = j\rangle$ is rotated by an infinitesimal angle $\varepsilon$ about y-axis. Withoug using the explicit form of the $d^{(j))}_{m'm}$ function, obtain an expression for the probability for a new rotated state to be found in the original state up to terms of order $\varepsilon^2$
**Solution:**
Let the given state be $|\alpha\rangle = |j,j\rangle$The rotation operator thorugh $Y$ axis is

$$\mathcal{D}_y(\varepsilon) = \exp\left(\frac{-iJ_y\varepsilon}{\hbar}\right) = 1 - \frac{iJ_y\varepsilon}{\hbar} - \frac{J_y^2\varepsilon^2}{2\hbar^2} + \dots$$

Writing $J_y = \frac{1}{2i}\left(J_+ - J_-\right)$ and expanding out the expression we get

$$\mathcal{D}_y(\varepsilon) = 1 - \frac{\varepsilon^2}{8\hbar^2}J_+J_-$$

So the rotetaed state is

$$\left|\alpha\right\rangle_R = \mathcal{D}_y(\varepsilon)\left|\alpha\right\rangle = 1 - \frac{\varepsilon^2}{8\hbar^2}J_+J_-\left|jj\right\rangle$$

The probability of finding the rotated state in the original state is given by $\left|\left\langle\alpha|\alpha\right\rangle_R\right|^2$ calculating thi

$$\left|\left\langle\alpha|\alpha\right\rangle_R\right|^2 = \left|\left\langle jj\right|1 - \frac{\varepsilon^2}{8\hbar^2}J_+J_-\left|jj\right\rangle\right|^2 = \left|\left\langle jj|jj\right\rangle - \frac{\varepsilon^2}{8\hbar^2}\left\langle jj\right|J_+J_-\left|jj\right\rangle\right|^2$$
$$= \left|1 - \sqrt{2j}\hbar\sqrt{2j}\hbar\right|^2 = \left|1 - \frac{\varepsilon^2 j}{4}\right|^2 \approx 1 - \frac{\varepsilon^2 j}{2}$$

This is the required probability in the order of $\varepsilon^2$. □

3. **(Sakurai 3.16)** Show that the orbital angular-momentum operator $\mathbf{L}$ commutes with both the operators $\mathbf{p}^2$ and $\mathbf{x}^2$
**Solution:**
The commutator of each component of $L$ with $\mathbf{p}^2$ are

$$\begin{aligned}\left[L_z, \mathbf{p}^2\right] &= \left[xp_y - yp_x, p_x^2 + p_y^2 + p_z^2\right] \\ &= \left[xp_y, p_x^2\right] - \left[yp_x, p_y^2\right] \\ &= \left(i\hbar\frac{\partial}{\partial p_x}p_x^2\right)p_y - \left(i\hbar\frac{\partial}{\partial p_y}p_y^2\right)p_x \\ &= 2i\hbar[p_x, p_y] \\ &= 0\end{aligned}$$

Similarly we can show that this is true for every component of the $\mathbf{L}$ hence it is proved for $\left[\mathbf{L}, \mathbf{p}^2\right]$.
Now for the commutation of $\mathbf{x}^2$ witht he operator $\mathbf{L}$

$$\begin{aligned}\left[L_z, \mathbf{x}^2\right] &= \left[xp_y - yp_x, p_x^2 + p_y^2 + p_z^2\right] \\ &= \left[xp_y, p_y^2\right] - \left[yp_x, p_x^2\right] \\ &= x\left(-i\hbar\frac{\partial}{\partial y}y^2\right) - y\left(-i\hbar\frac{\partial}{\partial x}x^2\right) \\ &= -2i\hbar[x, y] \\ &= 0\end{aligned}$$

Sincce this is true for the $L_z$ component it is also true for every toehr comopnent so that the vector commutatior $\left[\mathbf{L}, \mathbf{x}^2\right]$

□

4. **(Sakurai eq 3.6.11)** Prove the following
   (a)
$$\langle \mathbf{x}'|L_x|\alpha\rangle = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\langle \mathbf{x}'|\alpha\rangle$$
   (b)
$$\langle \mathbf{x}'|L_y|\alpha\rangle = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\langle \mathbf{x}'|\alpha\rangle$$
   (c)
$$\langle \mathbf{x}'|\mathbf{L}^2|\alpha\rangle = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\frac{\partial}{\partial\theta}\right)\right]\langle \mathbf{x}'|\alpha\rangle$$

**Solution:**
The angular momentum operator is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{P} = \mathbf{r} \times (-i\hbar\nabla) = (-i\hbar)r \times \nabla$$

These vectors in sperical coordinate system are

$$\mathbf{r} = r\hat{\mathbf{r}} + \theta\hat{\theta} + \phi\hat{\phi} \qquad \nabla = \hat{\mathbf{r}}\theta\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}$$

So teh corss product is

$$\mathbf{L} = (-i\hbar)\mathbf{r} \times \nabla = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\theta} & \hat{\phi} \\ r & \theta & \phi \\ r\frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\end{vmatrix} = (-\hbar)\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

Now the cartesian unit vectors in the spherical unit vectors are

$$\hat{\mathbf{x}} = \hat{\mathbf{r}} \sin\theta \cos\phi + \hat{\theta} \cos\theta \cos\phi - \hat{\phi} \sin\phi$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin\theta \sin\phi + \hat{\theta} \cos\theta \sin\phi + \hat{\phi} \sin\phi$$

Thus the angular momentum operator in the $L_x$ direction becomes

$$L_x = \hat{\mathbf{x}} \cdot \mathbf{L} = (-i\hbar) \left( -\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi} \right)$$

Thus

$$\langle \mathbf{x} | L_x | \alpha \rangle = -(-i\hbar) \left( -\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi} \right) \langle \mathbf{x} | \alpha \rangle$$

Similarly the operator $L_y$ is

$$L_y = \hat{\mathbf{y}} \cdot \mathbf{L} = (-i\hbar) \left( \cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi} \right)$$

Thus

$$\langle \mathbf{x} | L_y | \alpha \rangle = -(-i\hbar) \left( -\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi} \right) \langle \mathbf{x} | \alpha \rangle$$

Also the angular momentum squared opeator becomes

$$\begin{aligned}
\mathbf{L}^2 = \mathbf{L} \cdot \mathbf{L} &= \left[ (-\hbar) \left( \hat{\phi} \frac{\partial}{\partial\theta} - \hat{\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} \right) \right] \cdot \left[ (-\hbar) \left( \hat{\phi} \frac{\partial}{\partial\theta} - \hat{\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} \right) \right] \\
&= -\hbar^2 \left[ \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \\
&= -\hbar^2 \left[ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]
\end{aligned}$$

Thus we can write

$$\langle \mathbf{x} | \mathbf{L}^2 | \alpha \rangle = -\hbar^2 \left[ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] \langle \mathbf{x} | alpha \rangle$$

These are the required operator representation in spherical coordinate system. □