PHYS 517: Quantum Mechanics II

Homework #2

Prakash Gautam

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1. **(Sakurai 3.1)** Find the eigenvectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Suppose an electron is in spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. If S_y is measured, what is the probability of the result $\hbar/2$?
Solution:
Suppose the eignevalues of the matrix are λ. The characterstics equation for the matrix is

$$(0-\lambda)(0-\lambda) - (-i \cdot i) = 0 \qquad \Rightarrow \lambda = \pm 1$$

Let the eigenvector be $\begin{pmatrix} x \\ y \end{pmatrix}$. Then the eigenvector corresponding to $\lambda = 1$ we have

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -iy & = x \\ ix & = y \end{matrix} \quad \Rightarrow \quad \begin{matrix} x & = 1 \\ y & = i \end{matrix}$$

Normalizing this eivenvector we have the normalization factor $\sqrt{1^2+1^2} = \sqrt{2}$. So the required normalized eigenvector corresponding to $\lambda = 1$ is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Then the eigenvector corresponding to $\lambda = -1$ we have

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -iy & = -x \\ ix & = -y \end{matrix} \quad \Rightarrow \quad \begin{matrix} x & = & 1 \\ y & = & -i \end{matrix}$$

Normalizing this eivenvector we have the normalization factor $\sqrt{1^2+1^2} = \sqrt{2}$. So the required normalized eigenvector corresponding to $\lambda = -1$ is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

So the eigenvectors corresponding to each eigenvalues are

$$\lambda = 1 \quad \rightarrow \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \lambda = -1 \quad \rightarrow \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Let the arbitrary spin state be $|\gamma\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ such that its dual correspondence is $\langle\gamma| = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$. Since the matrix representation of the S_y operator is $\frac{\hbar}{2}\sigma_y$. The probability that the state be measure to be in S_y with eigenvalue $\frac{\hbar}{2}$ is

$$\langle\gamma|\hbar/2\sigma_2|\gamma\rangle = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \frac{i\hbar}{2} \left(-\beta\alpha^* + \beta^*\alpha\right)$$

So the probability of measuring the given state in $|S_y;+\rangle$ state is $\frac{i\hbar}{2}\left(\alpha\beta^* - \alpha^*\beta\right)$. □

2. **(Sakurai 3.2)** Find, by explicit construction using Pauli matrices, the eigenvalues for Hamiltonian

$$H = -\frac{2\mu}{\hbar}\mathbf{S}\cdot\mathbf{B}$$

for a spin $\frac{1}{2}$ particle in the presence a magnetic $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$.
Solution:
The hamiltonian operator in the given magnetic field as

$$H = -\frac{2\mu}{\hbar}\left(S_xB_x + S_yB_y + S_zB_z\right)$$

Since the spin operators S_x, S_y and S_z are the pauli matrices with a factor of $\hbar/2$ we can write the above expression as

$$\begin{aligned} H &= -\frac{2\mu}{\hbar}\frac{\hbar}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}B_x + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}B_y + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}B_z\right] \\ &= -\mu\begin{pmatrix}B_z & B_x - iB_y\\ B_x + iB_y & -B_z\end{pmatrix}\end{aligned}$$

The characterstics equation for the this matrix is

$$((B_z-\lambda)(-B_z-\lambda) - (B_x - iB_y)(B_x+iB_y)) = 0 \quad \Rightarrow \lambda^2 - B_z^2 - (B_x^2 + B_y^2) = 0 \quad \Rightarrow \lambda = \pm|\mathbf{B}|$$

So the eigenvalue of the Hamiltonian which is $-\mu$ times the matrix is $-\mu\cdot\lambda = \mp\mu|\mathbf{B}|$. □

3. **(Sakurai 3.3)** Consider 2×2 matrix defined by

$$U = \frac{a_0 + i\sigma\cdot\mathbf{a}}{a_0 - i\sigma\cdot\mathbf{a}}$$

where a_0 is a real number and $\mathbf{a}$ is a three-dimensional vector with real components.

(a) Prove that U is unitary and unimodular.
Solution:
Given matrix U and hermitian conjugate can be written as

$$U = \frac{a_0 + i\sum_j a_j\sigma_j}{a_0 - i\sum_j a_j\sigma_j} \qquad U^\dagger = \frac{a_0 - i\sum_j a_j\sigma_j^\dagger}{a_0 - i\sum_j a_j\sigma_j^\dagger}$$

Multiplying these two to check for unitarity

$$\begin{aligned} U^\dagger U &= \frac{a_0 - i\sum_j a_j\sigma_j^\dagger}{a_0 - i\sum_j a_j\sigma_j^\dagger}\cdot\frac{a_0 + i\sum_j a_j\sigma_j}{a_0 - i\sum_j a_j\sigma_j} \\ &= \frac{a_0^2 + ia_0\sum_j\sigma_ja_j - ia_0\sum_j\sigma_j^\dagger a_j + \sum_j\sum_k\sigma_j^\dagger a_j\sigma_ka_k}{a_0^2 - ia_0\sum_j\sigma_ja_j + ia_0\sum_j\sigma_j^\dagger a_j + \sum_j\sum_k\sigma_j^\dagger a_j\sigma_ka_k}\end{aligned}$$

Since each pauli matrices are Hermitian, for each i we have $\sigma_i^\dagger = \sigma_i$. This makes the numerator the exact same as the denominator. Thus they cancel out

$$U^\dagger U = \frac{a_0^2 + ia_0\sum_j\sigma_ja_j - ia_0\sum_j\sigma_ja_j + \sum_j\sum_k\sigma_ja_j\sigma_ka_k}{a_0^2 - ia_0\sum_j\sigma_ja_j + ia_0\sum_j\sigma_ja_j + \sum_j\sum_k\sigma_ja_j\sigma_ka_k} = 1$$

This shows that this matrix is unitary. Expanding out the matrix in terms of the pauli matrices we get

$$\det U = \frac{\begin{vmatrix}a_0 + ia_3 & ia_1 + a_2\\ ia_1 - a_2 & a_0 - ia_3\end{vmatrix}}{\begin{vmatrix}a_0 - ia_3 & -ia_1 + a_2\\ -ia_1 - a_2 & a_0 + ia_3\end{vmatrix}} = \frac{(a_0+ia_3)(a_0-ia_3) - (ia_1+a_2)(ia_1-a_2)}{(a_0-ia_3)(a_0+ia_3) - (-ia_1+a_2)(-ia_1-a_2)} = \frac{a_0^2+a_1^2+a_2^2+a_3^2}{a_0^2+a_1^2+a_2^2+a_3^2} = 1$$

This shows that the matrix is unimodular. □

(b) In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and the angle of rotation appropriate for U in terms of a_0, a_1, a_2 and a_3.

Solution:

The matrix can be rewritten as

$$U = \frac{1}{a_0^2+\mathbf{a}^2}\begin{pmatrix} a_0 - \mathbf{a}^2 + 2ia_0a_3 & 2a_0a_2 + 2ia_0a_1 \\ -2a_0a_2 + 2ia_0a_1 & a_0 - \mathbf{a}^2 - 2ia_0a_3 \end{pmatrix}$$

Since the most general unimodular matrix of the form $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ represent a rotaion through an angle ϕ through the direction $\hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$ related as

$$\mathrm{Re}(a) = \cos\left(\frac{\phi}{2}\right), \qquad \mathrm{Im}(a) = -n_z\sin\left(\frac{\phi}{2}\right) \tag{1}$$

$$\mathrm{Re}(b) = -n_y\sin\left(\frac{\phi}{2}\right), \qquad \mathrm{Im}(b) = -n_x\sin\left(\frac{\phi}{2}\right) \tag{2}$$

Making these comparision in this matrix we get

$$\cos\left(\frac{\phi}{2}\right) = \frac{a_0^2-\mathbf{a}^2}{a_0^2+\mathbf{a}^2} \qquad \Rightarrow \phi = 2\,\mathrm{acos}\left(\frac{a_0^2-\mathbf{a}^2}{a_0^2+\mathbf{a}^2}\right)$$

And similarly we get

$$n_x = -\frac{a_1}{|\mathbf{a}|} \qquad n_y = -\frac{a_2}{|\mathbf{a}|} \qquad n_z = -\frac{a_3}{|\mathbf{a}|}$$

This gives the rotation angle and the direction of rotation for this given unimodular matrix. □

4. **(Sakurai 3.9)** Consider a sequence of rotations represented by

$$\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \\ e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}$$

Solution:

Again this final matrix can be written as a coplex form as

$$\mathcal{D}^{1/2}(\alpha,\beta,\gamma) = \begin{bmatrix} \left(\cos\left(\frac{\alpha+\gamma}{2}\right) + i\sin\left(\frac{\alpha+\gamma}{2}\right)\right)\cos\left(\frac{\beta}{2}\right) & -\left(\cos\left(\frac{\alpha+\gamma}{2}\right) - i\sin\left(\frac{\alpha+\gamma}{2}\right)\right)\cos\left(\frac{\beta}{2}\right) \\ \left(\cos\left(\frac{\alpha+\gamma}{2}\right) + i\sin\left(\frac{\alpha+\gamma}{2}\right)\right)\cos\left(\frac{\beta}{2}\right) & -\left(\cos\left(\frac{\alpha+\gamma}{2}\right) - i\sin\left(\frac{\alpha+\gamma}{2}\right)\right)\cos\left(\frac{\beta}{2}\right) \end{bmatrix}$$

Let ϕ be the angle of rotation represented by this final rotation matrix. Using again the equations (1) we get

$$\cos\left(\frac{\phi}{2}\right) = \cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right) \qquad \Rightarrow \qquad \phi = 2\cos^{-1}\left[\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right]$$

This gives the angular rotation value for this matrix. The direction of rotation can similarly be found by using (1) to calculate the directions. □

5. **(Sakurai 3.15a)** Let $\mathbf{J}$ be angular momentum. Using the fact that J_x, J_y, J_z and $J_\pm \equiv J_x \pm J_y$ satisfy the usual angular-momentum commutation relations, prove

$$\mathbf{J}^2 = J_z^2 + J_+J_- - \hbar J_z$$

.

Solution:

Multiplying out J_+ and J_- we get

$$\begin{aligned} J_+J_- &= (J_x + iJ_y)(J_x - iJ_y) \\ &= J_x^2 - iJ_xJ_y + iJ_yJ_x + J_y^2 \\ &= J_x^2 - i[J_x, J_y] + J_y^2 \\ &= J_x^2 + J_y^2 - i(i\hbar J_z) \\ &= \mathbf{J}^2 - J_z^2 + \hbar J_z \end{aligned}$$

Rearranging above expression gives $\mathbf{J}^2 = J_z^2 + J_+J_- - \hbar J_z$ which completes the proof. □