PHYS : 516 Quantum Mechanics I

Homework #6

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1. Using the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right)S_z = \omega S_z$$

write the Heisenberg equation of motion for the time-dependent operators $S_x(t)$, $S_y(t)$ and $S_z(t)$. Solve them to obtain $S_{x,y,z}$ as functions of time

Solution:

We know the commutaion relation for spin operators $[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k$. And since the time derivative of the operator in Heisenberg picture is

$$\frac{\mathrm{d}}{\mathrm{d}t}(A) = \frac{1}{i\hbar}[A, H]$$

We can write the time derivative of the spin operators as

$$\frac{\mathrm{d}}{\mathrm{d}t}S_z = \frac{1}{i\hbar}[S_z, H] = \frac{1}{i\hbar}[S_z, \omega S_z] = \frac{1}{i\hbar}0 = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}S_x = \frac{1}{i\hbar}[S_x, H] = \frac{1}{i\hbar}[S_x, \omega S_z] = -\frac{\omega}{i\hbar}i\hbar S_y = -\omega S_y$$
$$\frac{\mathrm{d}}{\mathrm{d}t}S_y = \frac{1}{i\hbar}[S_y, H] = \frac{1}{i\hbar}[S_y, \omega S_z] = \omega\frac{1}{i\hbar}i\hbar S_x = \omega S_x$$

By similar fashion we can find the second time derivative of the operators as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}S_z = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}S_z\right) = \frac{\mathrm{d}}{\mathrm{d}t}(0) = 0$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}S_x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}S_x\right) = \frac{\mathrm{d}}{\mathrm{d}t}(-\omega S_y) = -\omega^2 S_x$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}S_y = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}S_y\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\omega S_x) = -\omega^2 S_y$$

Since the first time derivative of operator S_z is zero, it is constant over time. For $\frac{\partial^2}{\partial t^2}S_x = -\omega^2 S_x$ forms a Ordinary Second order differential equation in operator S_x. (Assuming derivatives are well defined for operators) We can write the solution as

$$S_x = Ae^{-i\omega t} \qquad\qquad S_y = Be^{-i\omega t}$$

Where A and B are arbitary constant (complex) numbers. ■

2. Consider a particle in one dimension whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x)$$

By calculating $[[H,x],x]$ prove

$$\sum_{a'} \left| \langle a''|x|a'\rangle \right|^2 (E_{a'} - E_{a''}) = \frac{\hbar^2}{2m},$$

where $|a'\rangle$ is an energy eigneket with eigenvalue $E_{a'}$

Solution:

Since x is Hermitian operator and $V(x)$ is pure function of x the commutator of x and $V(x)$ is zero i.e., $[x, V(x)] = 0$. By similar arguments the commutator of p and $\frac{p^2}{2m}$ is zero i.e., $\left[p, \frac{p^2}{2m}\right] = 0$ So we can calculate the commutator

$$[H,x] = \left[\frac{p^2}{2m} + V(x), x\right] = \frac{1}{2m}\left[p^2, x\right] = -i\hbar\frac{p}{m}$$

Also we can simplify the commutator as

$$[[H,x],x] = \left[i\hbar\frac{p}{m}, x\right] = -\frac{i\hbar}{m}[p,x] = -\frac{\hbar^2}{m}$$

the expectation value of the operator $[[H,x],x]$ can be calculated as

$$\begin{aligned}
\langle [[H,x],x] \rangle &= \left\langle a'' \middle| [[H,x],x] \middle| a'' \right\rangle \\
&= \left\langle a'' \middle| [Hx - xH, x] \middle| a'' \right\rangle \\
&= \left\langle a'' \middle| Hx^2 - xHx - xHx + x^2H \middle| a'' \right\rangle \\
&= \left\langle a'' \middle| Hx^2 \middle| a'' \right\rangle + \left\langle a'' \middle| x^2 H \middle| a'' \right\rangle - 2\left\langle a'' \middle| xHx \middle| a'' \right\rangle \\
&= E_{a''}\left\langle a'' \middle| x^2 \middle| a'' \right\rangle + E_{a''}\left\langle a'' \middle| x^2 \middle| a'' \right\rangle - 2\left\langle a'' \middle| xHx \middle| a'' \right\rangle \\
&= 2E_{a''}\left\langle a'' \middle| x^2 \middle| a'' \right\rangle - 2\left\langle a'' \middle| xHx \middle| a'' \right\rangle
\end{aligned}$$

Now the quantity $\langle a''|x^2|a''\rangle$ can be written as

$$\left\langle a'' \middle| x^2 \middle| a'' \right\rangle = \left\langle a'' \middle| xx \middle| a'' \right\rangle = \sum_{a'} \left\langle a'' \middle| x \middle| a' \right\rangle \left\langle a' \middle| x \middle| a'' \right\rangle = \sum_{a'} \left| \left\langle a'' \middle| x \middle| a' \right\rangle \right|^2$$

Similarly we can express $\langle a''|xHx|a''\rangle$ as

$$\left\langle a'' \middle| xHx \middle| a'' \right\rangle = \sum_{a'} \left\langle a'' \middle| xH \middle| a' \right\rangle \left\langle a' \middle| x \middle| a'' \right\rangle = \sum_{a'} E_{a'} \left\langle a'' \middle| x \middle| a' \right\rangle \left\langle a' \middle| x \middle| a'' \right\rangle = \sum_{a'} E_{a'} \left| \left\langle a'' \middle| x \middle| a' \right\rangle \right|^2$$

Finally these can be substitued to give

$$\langle [[H,x],x] \rangle = 2\sum_{a'} (E_{a'} - E_{a''}) \left| \left\langle a'' \middle| x \middle| a' \right\rangle \right|^2$$

But since we calcluated the epectation value to be $-\frac{\hbar^2}{m}$ we can write the expression

$$\sum_{a'} (E_{a'} - E_{a''}) \left| \left\langle a'' \middle| x \middle| a' \right\rangle \right|^2 = \frac{\hbar^2}{2m}$$

This is the rquired expression. ■