PHYS : 516 Quantum Mechanics I

Homework #5

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1. (a) Prove the following

   i. $\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$

   ii. $\langle\beta|x|\alpha\rangle = \int dp'\phi^*_\beta(p')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p'),$

   where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum-space wave functions.
   Solution:
   We know

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right);\ \text{and} \int\limits_{-\infty}^{\infty} e^{i(t-t')x}dx = 2\pi\delta(t-t')$$

With the help of these two relations we can simplify the quantity we want as

$$\begin{aligned}
\langle p'|x|\alpha\rangle &= \int dx'\,\langle p'|x|x'\rangle\,\langle x'|\alpha\rangle && \left(\because \int dx'\,|x'\rangle\langle x'| = 1\right)\\
&= \int x'\,\langle p'|x'\rangle\,\langle x'|\alpha\rangle\,dx' && (\because \langle p'|x|x'\rangle = x\,\langle p'|x'\rangle)\\
&= \int dp''\int x'\,\langle p'|x'\rangle\,\langle x'|p''\rangle\,\langle p''|\alpha\rangle\,dx' && \left(\because \int dp''\,|p''\rangle\langle p''| = 1\right)\\
&= \int dp''\int x'\frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)\cdot\frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{ip''x'}{\hbar}\right)\langle p''|\alpha\rangle\,dx'\\
&= \frac{1}{2\pi\hbar}\int dp''\int x'\exp\left(\frac{i(p'-p'')x'}{\hbar}\right)\langle p''|\alpha\rangle\,dx'
\end{aligned}$$

We can use integral under differential sign to evaluate the dx' integral as

$$\frac{d}{dp'}\int\exp(i(p'-p'')x')dx' = \int x'\exp(i(p'-p'')x')dx'$$

Using ths in the dx' integral above we get

$$\begin{aligned}
&= \frac{1}{2\pi\hbar}\int dp''\frac{\hbar^2}{-i}\frac{\partial}{\partial p'}\int\exp\left(\frac{i(p'-p'')x'}{\hbar}\right)\langle p''|\alpha\rangle\,dx'\\
&= \frac{1}{2\pi\hbar}\int dp''\frac{\hbar^2}{-i}\frac{\partial}{\partial p'}2\pi\delta(p'-p'')\,\langle p''|\alpha\rangle\\
&= i\hbar\,\langle p'|\alpha\rangle && \left(\because \int f(x)\delta(x-x')dx = f(x')\right)
\end{aligned}$$

This gives us the requied result.

$$\langle\beta|x|\alpha\rangle = \int dp' \langle\beta|p'\rangle \langle p'|x|\alpha\rangle \tag{1}$$

The result above is $\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$ Substuting this in (**??**) we get

$$\langle\beta|x|\alpha\rangle = \int dp' \langle\beta|p'\rangle i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$$

Writing $\langle\beta|p'\rangle = \phi^*_\beta(p')$ and $\langle p'|\alpha\rangle = \phi_\alpha(p')$ we get

$$\langle\beta|x|\alpha\rangle = \int dp' \phi^*_\beta(p') i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$$

This is the requied expression. ■

(b) What is the physical significance of

$$\exp\left(\frac{ix\Xi}{\hbar}\right)$$

where x is the position operator and Ξ is some number with the unit of momentum? Justify your answer.
Solution:
In the position eigenbasis the position translation operator is $\sqcup(l) = \exp\left(\frac{ipl}{\hbar}\right)$ where l is a constant of unit of lenght and p is the momentum operator.
We have here the roles of operator x and p changed and l and Ξ changed. Which suggests that this operator function can works as a momentum translation operator in momentum eigenbasis. ■

2. If the Hamiltonian H is given as

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}|1\rangle\langle 2|$$

What principle is violated? Illustrate your point by explicitly attempting to solve the most general time-dependent problem using an illegal Hamiltonian of this kind. (Assume $H_{11} = H_{22} = 0$ for simplicity.)
Solution:
For a operator to be a valid Hamiltonian it has to be a Hermitian operator. We can check if this is a Hermitian operator.

$$H^\dagger = H^*_{11}|1\rangle\langle 1| + H^*_{22}|2\rangle\langle 2| + H^*_{12}|1\rangle\langle 2| = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}|1\rangle\langle 2|$$

Since $H^\dagger \neq H$ the given hamiltonian is clearly not Hermitian. So this operator the energy eigenkets won't be real. Also, the time translation operator $\mathcal{U}(t) = \exp\left(-\frac{iHt}{\hbar}\right)$ will not be unitary which would make the time evolved states not conserve the inner product so, it violates the principle of probability violation.

Setting $H_{11} = H_{22} = 0$ the Hamiltonian becomes $H = H_{12}|1\rangle\langle 2|$. Lets check the unitary property of the unitary operator

$$\mathcal{U}^\dagger(t)\mathcal{U}(t) = \exp\left(\frac{iH^\dagger t}{\hbar}\right)\cdot\exp\left(-\frac{iHt}{\hbar}\right) = \exp\left(\frac{i(H^\dagger - H)t}{\hbar}\right)$$

For the operator to remain unitary, the exponential should be zero but since $H^\dagger \neq H$ the exponent will be nonzero and it violates the principle that the time evolution operator si unitary. ■

3. Let $|a'\rangle$ and $|a''\rangle$ be eigenstates of a Hermitian operator A with eigenvalues a' and a'', respectively ($a' \neq a''$). The Hamiltonian operator is given by

$$H = |a'\rangle \delta \langle a''| + |a''\rangle \delta \langle a'|$$

where δ is just a real number.

(a) Clearly, $|a'\rangle$ and $|a''\rangle$ are not eigenstates of teh Hamiltonian. Write down the eigenstates of the Hamiltonian. WHat are their energy Eigenvalues?

Solution:

Let the energy eigenket of this hamiltonian operator be $|\alpha\rangle = p|a'\rangle + q|a''\rangle$. And E be the energy eigen values. So operating by H on this state leads to

$$\begin{aligned} H|\alpha\rangle &= (|a'\rangle \delta \langle a''| + |a''\rangle \delta \langle a'|)(p|a'\rangle + q|a''\rangle) \\ &= \delta q|a'\rangle + \delta p|a''\rangle \end{aligned}$$

If this is to be the energy eigenstate then it should equal $E|\alpha\rangle = Ep|a'\rangle + Eq|a''\rangle$. Since $|a'\rangle$ and $|a''\rangle$ are orthogonal states, the coefficient comparision leads to

$$\begin{aligned} Ep = \delta q; &\qquad \Rightarrow p = \frac{\delta q}{E} \\ Eq = \delta p; &\qquad \Rightarrow Eq = \delta\frac{\delta q}{E}; \qquad \Rightarrow E = \pm\delta \end{aligned}$$

So the energy eigenvalues are $E = \pm\delta$. Also since we require the eigenstate be normalized we require $p^2 + q^2 = 1$. This results in

$$\frac{\delta^2 q^2}{E^2} + q^2 = 1; \qquad \Rightarrow p = \frac{1}{\sqrt{2}}, \quad q = \pm\frac{1}{\sqrt{2}}$$

So the requried energy eigenkets are

$$|\alpha_+\rangle = \frac{1}{\sqrt{2}}(|a'\rangle + |a''\rangle); \qquad |\alpha_-\rangle = \frac{1}{\sqrt{2}}(|a'\rangle - |a''\rangle) \tag{2}$$

Where $|\alpha_+\rangle$ is the eigenket corresponding to eigenvalue $+\delta$ and $|\alpha_-\rangle$ is the eigenket corresponding to eigenvalue $-\delta$ ■

(b) Suppose the system is known to be in the state $|a'\rangle$ at $t = 0$. Write down the state vector of Schrodinger picture for $t > 0$.

Solution:

The time evolution operator is $\mathcal{U}(t) = \exp\left(-\frac{iHt}{\hbar}\right)$. Since $|a'\rangle$ are not the eiergy eigenkets, we can write them in terms of the eigenkets of Hamiltonian operator. From (**??**) we can add and subtract the two energy eigenkets to find

$$|a'\rangle = \frac{1}{\sqrt{2}}(|\alpha_+\rangle + |\alpha_-\rangle) \qquad\qquad |a''\rangle = \frac{1}{\sqrt{2}}(|\alpha_+\rangle - |\alpha_-\rangle)$$

Application of time evolution operator to $|a'\rangle$ leads to

$$\mathcal{U}(t)|a'\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|a'\rangle = \exp\left(-\frac{iHt}{\hbar}\right)\frac{1}{\sqrt{2}}(|\alpha_+\rangle + |\alpha_-\rangle = \frac{1}{\sqrt{2}}e^{-i\delta\frac{t}{\hbar}}|\alpha_+\rangle + \frac{1}{\sqrt{2}}e^{i\delta\frac{t}{\hbar}}|\alpha_-\rangle$$

Again the application of (**??**) we can convert back to the basis states given

$$\mathcal{U}(t)|a'\rangle = \frac{1}{2}e^{-i\delta\frac{t}{\hbar}}(|a'\rangle + |a''\rangle) + \frac{1}{2}e^{i\delta\frac{t}{\hbar}}(|a'\rangle - |a''\rangle) = \frac{1}{2}(\underbrace{e^{-i\frac{\delta t}{\hbar}} + e^{i\frac{\delta t}{\hbar}}}_{2\cos\left(\frac{\delta t}{\hbar}\right)})|a'\rangle + \frac{1}{2}(\underbrace{e^{-i\frac{\delta t}{\hbar}} - e^{i\frac{\delta t}{\hbar}}}_{2i\sin\left(\frac{\delta t}{\hbar}\right)})|a''\rangle$$

Euler identity can be used to convert the complex exponentials to sines and cosies, which give

$$\mathcal{U}(t)\left|a'\right\rangle = \cos\left(\frac{\delta t}{\hbar}\right)\left|a'\right\rangle + i\sin\left(\frac{\delta t}{\hbar}\right)\left|a''\right\rangle \tag{3}$$

This gives the time evolution of state $|a'\rangle$ under this hamiltonian. ■

(c) What is the probability for finding the system in $|a''\rangle$ for $t > 0$ if the system is known to be in the state $|a'\rangle$ at $t = 0$?
Solution:
The probability of finting the system knon to be in $|a'\rangle$ at a later time $t > 0$ is given by $|\langle a''|\mathcal{U}(t)|a'\rangle|^2$ which can be evauated using (**??**)

$$P = |\langle a''|\mathcal{U}(t)|a'\rangle|^2 = \left|\langle a''|\left[\cos\left(\frac{\delta t}{\hbar}\right)|a'\rangle + i\sin\left(\frac{\delta t}{\hbar}\right)|a''\rangle\right]\right|^2 = \left|i\sin\left(\frac{\delta t}{\hbar}\right)\right|^2 = \sin^2\left(\frac{\delta t}{\hbar}\right)$$

So the probability of finding the $|a'\rangle$ to be at $|a''\rangle$ at a later time is the oscillating function. The physical situation corresponding to this problem is a Neutrino oscillation. ■

4. Show

$$\langle p'|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\int_{-\infty}^{\infty} dx' \exp\left(\frac{-ip'x'}{\hbar} + ikx' - \frac{x'^2}{2d^2}\right) = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right].$$

Solution:
Considering the factor inside the exponential

$$\frac{-ip'x'}{\hbar} + ikx' - \frac{x'^2}{2d^2} = -\frac{1}{2d^2}\left(x'^2 - 2d^2\left(ik - \frac{ip'x}{\hbar}\right)x'\right)$$

If we let the constant terms $t = d^2\left(ik - \frac{ip'x}{\hbar}\right)$ then in the exponential we get

$$\frac{-1}{2d^2}\left(x'^2 - 2tx'\right) \xrightarrow{\text{Completion of square}} \frac{-1}{2d^2}\left((x'-t)^2 - t^2\right)$$

With this the integral becomes

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x'^2}{2d^2}\right)\cdot\exp\left(\frac{t^2}{2d^2}\right)dx' = \exp\left(-\frac{t^2}{2d^2}\right)\int_{-\infty}^{\infty}\exp\left\{-\left(\frac{x'}{\sqrt{2}d}\right)^2\right\}dx'$$

This integral is a standard gamma function whose value is

$$\int_{-\infty}^{\infty}\exp\left\{-\left(\frac{x'}{\sqrt{2}d}\right)^2\right\}dx' = 2\int_{0}^{\infty}\exp\left\{-\left(\frac{x'}{\sqrt{2}d}\right)^2\right\}dx' = \frac{\sqrt{\pi}}{2}\cdot 2\sqrt{2}d$$

Using this in our original equation we get

$$\langle p'|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\exp\left(-\frac{t^2}{2d^2}\right)\sqrt{2\pi}d$$

We can substitute back the variable t back to get

$$\langle p'|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\exp\left(-\frac{t^2}{2d^2}\right)\sqrt{2\pi}d = \frac{1}{\sqrt{\hbar}}\left(\frac{\sqrt{d}}{\pi^{1/4}}\right)\exp\left(-\frac{d^4(ik - i\frac{p'x'}{\hbar})^2}{2d^2}\right)$$
$$= \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right].$$

Which is the required solution ■