PHYS : 502 Quantum Mechanics I

Homework #3

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1. Using the rules of bra-ket algebra, prove or evaluate the following:

   (a) $\text{tr}(XY) = \text{tr}(YX)$, where X and Y are operators
   Solution:
   The definition of trace of an operator is $tr(A) = \sum_{a'} \langle a'|A|a'\rangle$. Using this definition for operator XY we get

$$
\begin{aligned}
\text{tr}(XY) &= \sum_{a'} \langle a'|XY|a'\rangle && (\text{ Definition}) \\
&= \sum_{a'}\sum_{a''} \langle a'|X|a''\rangle\, \langle a''|Y|a'\rangle && (\sum_{a''} |a''\rangle\langle a''| = 1) \\
&= \sum_{a'}\sum_{a''} \langle a''|Y|a'\rangle\, \langle a'|X|a''\rangle && (\text{Complex number commute}) \\
&= \sum_{a''} \langle a''|YX|a''\rangle && (\sum_{a'} |a'\rangle\langle a'| = 1) \\
&= \text{tr}(YX) && (\text{By definition })
\end{aligned}
$$

   Thus $\text{tr}(XY) = \text{tr}(YX)$ as required ■

   (b) $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators.
   Solution:

   Let $|\alpha\rangle$ be any arbitrary ket.

$$\text{Let } Y\,|\alpha\rangle = |\gamma\rangle \qquad \leftarrow DC \rightarrow \qquad \langle\alpha|\,Y^\dagger = \langle\gamma|$$

   Using this fact and operating the arbitrary $|\alpha\rangle$ by the operator XY we get,

$$
\begin{aligned}
XY\,|\alpha\rangle &= X\,|\gamma\rangle && (\because Y\,|\alpha\rangle = |\gamma\rangle \text{ by assumption}) \\
\langle\alpha|\,(XY)^\dagger &= \langle\gamma|\,X^\dagger && (\because \text{ Taking DC on both sides}) \\
\langle\alpha|\,(XY)^\dagger &= \langle\alpha|\,Y^\dagger X^\dagger && (\because \langle\gamma| = \langle\alpha|\,Y^\dagger\ )
\end{aligned}
$$

   Which implies $(XY)^\dagger = X^\dagger Y^\dagger$ ■

   (c) $\exp(if(A)) = ?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known.
   Solution:
   Assuming the function can be written as $e^X = 1 + f(X) + \frac{f^2(X)}{2!} + \frac{f^3(X)}{3!} + \cdots$, where X is an operator in the ket space. We have

$$e^{if(A)} = \sum_{a'} e^{if(A)}\,|a'\rangle\langle a'| \qquad \left(\because \sum_{a'} |a'\rangle\langle a'| = 1\right)$$

Here $|a'\rangle$ are the eigenkets of the operator A as it is given to be a Hermitian operator. Using the expansion for $e^{if(A)}$ we get,

$$
\begin{aligned}
e^{if(A)} &= \sum_{a'} \left(1 + f(A) + \frac{f^2(A)}{2!} + \frac{f^3(A)}{3!} + \cdots\right) |a'\rangle\langle a'| && \left(\because \sum_{a'} |a'\rangle\langle a'| = 1\right) \\
&= \sum_{a'} \left(|a'\rangle + f(A)\,|a'\rangle + \frac{1}{2!} f^2(A)\,|a'\rangle + \cdots\right) \langle a'| && (\because X(|\alpha\rangle\langle\beta|) = (X\,|\alpha\rangle)\,\langle\beta|) \\
&= \sum_{a'} \left(|a'\rangle + f(a')\,|a'\rangle + \frac{1}{2!} f^2(a')\,|a'\rangle + \cdots\right) \langle a'| && (\because f(X)\,|a'\rangle = f(a')\,|a'\rangle \text{ for Hermitian } X) \\
&= \sum_{a'} \left(1 + f(a') + \frac{1}{2!} f^2(a') + \cdots\right) |a'\rangle\,\langle a'| && (\because (a\,|\alpha\rangle)\,\langle\beta| = a(|\alpha\rangle\langle\beta|)) \\
&= \sum_{a'} e^{f(a')}\,|a'\rangle\langle a'|
\end{aligned}
$$

Which is the required form for the operator $e^{f(A)}$. ■

2. A spin 1/2 system is known tobe in an eigenstate of $\mathbf{S}\cdot\hat{\mathbf{n}}$ with eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector lying in the xz-plane that makes and angle γ with the positive z-axis.

   (a) Suppose S_x is measured. What is the probability of getting $\hbar/2$
   Solution:
   For a two state system the general stae of system can be represented as $|\hat{\mathbf{n}};+\rangle = \cos\frac{\beta}{2}\,|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}\,|-\rangle$, where α is the polar angle and β is the azimuthal angle. For this problem the polar angle is $\alpha = 0$ and azimuthal angle is $\beta = \gamma$. So the given system and $|S_x;+\rangle$ states are

$$|\hat{\mathbf{n}};+\rangle = \sin\frac{\gamma}{2}\,|+\rangle + \cos\frac{\gamma}{2}\,|-\rangle\,; \quad |S_x;+\rangle = \frac{1}{\sqrt{2}}\,|+\rangle + \frac{1}{\sqrt{2}}\,|-\rangle$$

   Since by definition the probablility of measuring any state that is known to be in $|beta\rangle$ in a state $|\alpha\rangle$ is given by $|\langle\alpha|\beta\rangle|^2$. So the probability of measuring $|S_x;+\rangle$ state when the system is known to be in $|\hat{\mathbf{n}};+\rangle$ stae is

$$
\begin{aligned}
|\langle S_x;+|\hat{\mathbf{n}};+\rangle|^2 &= \left|\left(\frac{1}{\sqrt{2}}\,\langle +| + \frac{1}{\sqrt{2}}\,\langle -|\right)\left(\sin\frac{\gamma}{2}\,|+\rangle + \cos\frac{\gamma}{2}\,|-\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\sin\frac{\gamma}{2} + \frac{1}{\sqrt{2}}\cos\frac{\gamma}{2}\right|^2 \\
&= \frac{1}{2}\sin^2\frac{\gamma}{2} + 2\frac{1}{\sqrt{2}}\sin\frac{\gamma}{2}\frac{1}{\sqrt{2}}\cos\frac{\gamma}{2} + \frac{1}{2}\cos^2\frac{\gamma}{2} \\
&= \frac{1}{2}(1 + \sin\gamma)
\end{aligned}
$$

   So the probability of measuring the $|\hat{\mathbf{n}}\rangle$ state in $|S_x;+\rangle$ state is $(1+\sin\gamma)/2$. ■

   (b) Evaluate the dispersion in S_x –that is $\left\langle (S_x - \left\langle {S_x}^2 \right\rangle) \right\rangle$
   Solution:
   The S_x operator is $S_x = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)$. The result of S_x state operated on the system at $|\hat{\mathbf{n}}\rangle$ is

$$S_x\,|\hat{\mathbf{n}}\rangle = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)(\sin\frac{\gamma}{2}\,|+\rangle + \cos\frac{\gamma}{2}\,|-\rangle) = \frac{\hbar}{2}\cos\frac{\gamma}{2}\,|+\rangle + \frac{\hbar}{2}\sin\frac{\gamma}{2}\,|-\rangle$$

And the dual correspondence of the state $|\hat{\mathbf{n}}\rangle$ is $\langle\hat{\mathbf{n}}| = \sin\frac{\gamma}{2}\langle +| + \cos\frac{\gamma}{2}\langle -|$. So the expectation value of S_x is

$$\langle S_x\rangle = \langle\hat{\mathbf{n}}|S_x|\hat{\mathbf{n}}\rangle = \Big(\sin\frac{\gamma}{2}\langle +| + \cos\frac{\gamma}{2}\langle -|\Big)\Big(\frac{\hbar}{2}\cos\frac{\gamma}{2}|+\rangle + \frac{\hbar}{2}\sin\frac{\gamma}{2}|-\rangle\Big) = \frac{\hbar}{2}\Big(2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\Big) = \frac{\hbar}{2}\sin\gamma$$

Also the expectation value of operator S_x^2 is

$$\begin{aligned}\langle S_x^2\rangle = \langle\hat{\mathbf{n}}|S_xS_x|\hat{\mathbf{n}}\rangle &= \Big(\sin\frac{\gamma}{2}\langle +| + \cos\frac{\gamma}{2}\langle -|\Big)\Big(\frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)\Big)\Big(\frac{\hbar}{2}\cos\frac{\gamma}{2}|+\rangle + \frac{\hbar}{2}\sin\frac{\gamma}{2}|-\rangle\Big)\\ &= \Big(\sin\frac{\gamma}{2}\langle +| + \cos\frac{\gamma}{2}\langle -|\Big)\Big(\frac{\hbar^2}{4}\Big(\sin\frac{\gamma}{2}|+\rangle + \cos\frac{\gamma}{2}|-\rangle\Big)\Big)\\ &= \frac{\hbar^2}{4}\Big(\sin^2\frac{\gamma}{2} + \cos^2\frac{\gamma}{2}\Big)\\ &= \frac{\hbar^2}{4}\end{aligned}$$

Now the dispersion by definition is

$$\langle\Delta S_x^2\rangle \equiv \langle S_x^2\rangle - (\langle S_x\rangle)^2 = \frac{\hbar^2}{4} - \Big(\frac{\hbar}{2}\sin\gamma\Big)^2 = \frac{\hbar^2}{4}\big(1-\sin^2\gamma\big) = \frac{\hbar^2}{4}\cos^2\gamma$$

Which gives the dispersion in measurement of S_x of the system in $|\hat{\mathbf{n}}\rangle$. ■

3. Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent withthe general relation $U = \sum_r \left|b^{(r)}\right\rangle\!\left\langle a^{(r)}\right|$

   Solution:
   The states $|S_x;\pm\rangle$ in the $|S_z;\pm\rangle \equiv |\pm\rangle$ state is given by $|S_x;\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$. Since we know the transformation matrix form is

$$\begin{bmatrix}\langle S_x;+|+\rangle & \langle S_x;+|-\rangle\\ \langle S_x;-|+\rangle & \langle S_x;-|-\rangle\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}(\langle +| + \langle -|)|+\rangle & (\langle +| + \langle -|)|-\rangle\\ (\langle +| - \langle -|)|+\rangle & (\langle +| - \langle -|)|-\rangle\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

   Let $|p\rangle = a|+\rangle + a|-\rangle$ in the old S_z basis. such that $a = \langle +|p\rangle$ and $b = \langle -|p\rangle$. This ket is transformed into

$$\begin{aligned}Mp = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} &\equiv \frac{1}{\sqrt{2}}(a+b)|+\rangle + \frac{1}{\sqrt{2}}(a-b)|-\rangle && (1)\\ &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)a + \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)b && (2)\\ &= \frac{1}{\sqrt{2}}((|+\rangle + |-\rangle)\langle +|p\rangle + \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\langle -|p\rangle) && (3)\\ &= (|S_x;+\rangle\langle +| + |S_x;-\rangle\langle -|)|p\rangle && (4)\end{aligned}$$

   Which is in the form of $\sum |b^r\rangle\langle a^r|$. ■

4. Prove that $\langle\mathbf{x}\rangle \to \langle\mathbf{x}\rangle + d\mathbf{x}'$, $\langle\mathbf{p}\rangle \to \langle\mathbf{p}\rangle$ under infinitesimal translation.
   Solution:
   Since given

$$[\mathbf{x}, \mathcal{T}(d\mathbf{x})] = d\mathbf{x}; \Rightarrow \mathbf{x}\mathcal{T}(d\mathbf{x}) - \mathcal{T}(d\mathbf{x})\mathbf{x} = d\mathbf{x}; \qquad \mathbf{x}\mathcal{T}(d\mathbf{x}) = d\mathbf{x} + \mathcal{T}(d\mathbf{x})\mathbf{x}$$

Let the state of system under translation be $|\beta\rangle = \mathcal{T}(d\mathbf{x})\,|\alpha\rangle$, thus $\langle\beta| = \langle\alpha|\,\mathcal{T}^\dagger(d\mathbf{x})$. Now the expectation value of system before translation is $\langle\mathbf{x}\rangle = \langle\alpha|\mathbf{x}|\alpha\rangle$. The expectation value after translation is

$$\begin{aligned}
\langle\mathbf{x}\rangle &= \langle\beta|\mathbf{x}|\beta\rangle \\
&= \langle\alpha|\mathcal{T}^\dagger(d\mathbf{x})\mathbf{x}\mathcal{T}(d\mathbf{x})|\alpha\rangle \\
&= \langle\alpha|\mathcal{T}^\dagger(d\mathbf{x})(d\mathbf{x} + \mathcal{T}(d\mathbf{x})\mathbf{x})|\alpha\rangle \\
&= \langle\alpha|\mathcal{T}^\dagger(d\mathbf{x}) + \mathcal{T}^\dagger(d\mathbf{x})\mathcal{T}(d\mathbf{x})\mathbf{x}|\alpha\rangle \\
&= \langle\alpha|\mathcal{T}^\dagger(d\mathbf{x}) + \mathbf{x}|\alpha\rangle \\
&= \langle\alpha|\mathcal{T}^\dagger(d\mathbf{x})|\alpha\rangle + \langle\alpha|\mathbf{x}|\alpha\rangle \\
&= d\mathbf{x} + \langle\mathbf{x}\rangle
\end{aligned}$$

So the expectation value of position after translation is $\langle\mathbf{x}\rangle + d\mathbf{x}$.

Similarly for momentum

$|\beta\rangle = \mathcal{T}(d\mathbf{x})\,|\alpha\rangle$, thus $\langle\beta| = \langle\alpha|\,\mathcal{T}^\dagger(d\mathbf{x})$. Now the expectation value of momentum before translation is $\langle\mathbf{p}\rangle = \langle\alpha|\mathbf{p}|\alpha\rangle$. The expectation value after translation is

$$\begin{aligned}
\langle\mathbf{p}\rangle &= \langle\beta|\mathbf{p}|\beta\rangle \\
&= \langle\alpha|\mathcal{T}^\dagger(d\mathbf{x})\mathbf{p}\mathcal{T}(d\mathbf{x})|\alpha\rangle \\
&= \langle\alpha|\mathcal{T}^\dagger(d\mathbf{x})(0 + \mathcal{T}(d\mathbf{x})\mathbf{p})|\alpha\rangle \\
&= \langle\alpha|\mathcal{T}^\dagger(d\mathbf{x})\mathcal{T}(d\mathbf{x})\mathbf{p}|\alpha\rangle \\
&= \langle\alpha|\mathbf{p}|\alpha\rangle
\end{aligned}$$

So the expectation value of system after translation is still $\langle\mathbf{p}\rangle$. ■