PHYS :516 Quantum Mechanics I

Homework #2

Prakash Gautam

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1. A two state system is is characterized by a Hamiltonian $H_{11}|1\rangle\langle 1| + H_{12}\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right) + H_{22}|2\rangle\langle 2|$ where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and the corresponding energy eigenvalues.
   Solution:
   Let the energy eigenket be $|E\rangle = p|1\rangle + q|2\rangle$ and the eigenvalues be λ. OPerating thsi state by the given Hamiltonian Operator we get

$$
\begin{aligned}
H\,|E\rangle &= H_{11}|1\rangle\langle 1| + H_{12}\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right) + H_{22}|2\rangle\langle 2|(p|1\rangle + q|2\rangle) \\
&= H_{11}p\,\langle 1\,|1\rangle|1\rangle + H_{11}q\,\langle 1\,|2\rangle|1\rangle + H_{12}p\,\langle 1\,|1\rangle|2\rangle + H_{12}p\,\langle 2\,|1\rangle|1\rangle + H_{12}q\,\langle 1\,|2\rangle|2\rangle \\
&\quad + H_{12}q\,\langle 2\,|2\rangle|1\rangle + H_{22}p\,\langle 2\,|1\rangle|2\rangle + H_{22}q\,\langle 2\,|2\rangle|2\rangle \\
&= H_{11}p|1\rangle + H_{12}p|2\rangle + H_{12}q|1\rangle + H_{22}q|2\rangle \\
&= \left(H_{11}p + H_{12}q\right)|1\rangle + \left(H_{12}p + H_{22}q\right)|2\rangle
\end{aligned}
$$

   Since by assumption λ is the eigenvalue of this state we have $H\,|E\rangle = \lambda\,|E\rangle$ which gives

$$\lambda p\,|1\rangle + \lambda q\,|2\rangle = \left(H_{11}p + H_{12}q\right)|1\rangle + \left(H_{12}p + H_{22}q\right)|2\rangle$$

   Comparing the coefficient of each independent we get

$$\lambda p = \left(H_{11}p + H_{12}q\right); \qquad \lambda q = \left(H_{12}p + H_{22}q\right)$$

$$\Rightarrow (\lambda - H_{11})p - H_{12}q = 0; \quad p = \frac{H_12}{\lambda - H_{11}}q$$

$$H_{12}p + (H_{22} - \lambda)q = 0; \quad \Rightarrow H_{12}\left(\frac{H_{12}}{\lambda - H_{11}}\right)q + (H_{22} - \lambda)q = 0;$$

   Solving this for λ we get

$$\lambda = \frac{1}{2}(H_{11} + H_{22}) \pm \frac{1}{2}\sqrt{H_{11}^2 - 2H_{11}H_{22} + 4H_{12}^2 + H_{22}^2}$$

   These are the required eigenvalues of the given operator. This eigenvalues can be plugged back into the given equation to get the values of p and q.

$$q = 1; \qquad p = \frac{H_{12}}{\frac{H_{22} - H_{11}}{2} \pm \frac{1}{2}\sqrt{H_{11}^2 - 2H_{11}H_{22} + 4H_{12}^2 + H_{22}^2}}$$

So the required eigenstates are

$$|E\rangle = \left(\frac{H_{12}}{\frac{H_{22}-H_{11}}{2} \pm \frac{1}{2}\sqrt{H_{11}^2 - 2H_{11}H_{22} + 4H_{12}^2 + H_{22}^2}} \right) |1\rangle + |2\rangle$$

The above eigenstae can be normalized if required to get the Energy eigenket. ■

2a) Compute $\langle (\Delta S_x)^2 \rangle \equiv \langle S_x^2 \rangle - \langle S_x \rangle^2$ where the expectation value is taken for the S_z+ state. Using your result check the generalized uncertainity relation

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2$$

with $A \to S_x, B \to S_y$.

Solution:

Let $|+\rangle$ represent the $|S_z;+\rangle$ state. Then the expectation value of S_x for $|S_z;+\rangle$ can be calculated as

$$S_z = \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right); \qquad S_y = \frac{i\hbar}{2}\left(-|+\rangle\langle -| + |-\rangle\langle +|\right); \qquad S_x = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right);$$

$$S_x|+\rangle = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle|+\rangle\right)|+\rangle = \frac{\hbar}{2}|-\rangle; \qquad S_x|-\rangle = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)|-\rangle = \frac{\hbar}{2}|+\rangle;$$

$$S_y|+\rangle = \frac{i\hbar}{2}\left(-|+\rangle\langle -| + |-\rangle\langle +|\right); |+\rangle = \frac{i\hbar}{2}|-\rangle; \qquad S_y|-\rangle = \frac{i\hbar}{2}\left(-|+\rangle\langle -| + |-\rangle\langle +|\right); |-\rangle = -\frac{i\hbar}{2}|+\rangle;$$

So the expectation values are

$$\begin{aligned}
\langle S_x \rangle &= \langle +|S_x|+\rangle = \langle +|\frac{\hbar}{2}|-\rangle = \frac{\hbar}{2}\langle +|-\rangle = 0 \\
\langle S_y \rangle &= \langle +|S_y|+\rangle = \langle +|\frac{i\hbar}{2}|-\rangle = -i\frac{\hbar}{2}\langle +|-\rangle = 0 \\
\langle S_x^2 \rangle &= \langle +|S_x^2|+\rangle = \langle +|S_xS_x|+\rangle = \langle +|S_x\frac{\hbar}{2}|-\rangle = \frac{\hbar}{2}\langle +|\frac{\hbar}{2}|+\rangle = \frac{\hbar^2}{4}\langle +|-\rangle = \frac{\hbar^2}{4} \\
\langle S_y^2 \rangle &= \langle +|S_y^2|+\rangle = \langle +|S_yS_y|+\rangle = \langle +|S_yi\frac{\hbar}{2}|-\rangle = \frac{i\hbar}{2}\langle +|\frac{-i\hbar}{2}|+\rangle = -i^2\frac{\hbar^2}{4}\langle +|-\rangle = \frac{\hbar^2}{4}
\end{aligned}$$

Since $[S_x, S_y] = i\hbar S_z$ and $|\langle [S_x, S_y] \rangle|^2 = \langle [S_x, S_y] \rangle \langle [S_x, S_y] \rangle^*$ we can write

$$\langle [S_x, S_y] \rangle = \langle i\hbar S_z \rangle = i\hbar\langle +|S_z|+\rangle = i\hbar\langle +|\frac{\hbar}{2}|+\rangle = i\frac{\hbar^2}{2}; \qquad \langle [S_x, S_y] \rangle^* = -i\frac{\hbar^2}{2};$$

The dispersion in S_x and S_y can be calculated as

$$\langle (\Delta S_x)^2 \rangle \equiv \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}; \qquad \langle (\Delta S_x)^2 \rangle \equiv \langle S_x^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4};$$

Thus finally

$$\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle \geq \frac{1}{4} |\langle [S_x, S_y] \rangle|^2$$
$$\frac{\hbar^2}{4} \cdot \frac{\hbar^2}{4} \geq \frac{1}{4} \left(i\frac{\hbar^2}{2} \right) \left(-i\frac{\hbar^2}{2} \right)$$
$$\frac{\hbar^4}{16} \geq \frac{\hbar^4}{16}$$

Which is true as required. ■

(b) Check the uncertainity relation with $A \to S_x, B \to S_y$ for the S_x+ State

3. Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the unertainity product $\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle$. Verify explicitly that the linear combination you found, the uncertainty relation for S_x and S_y is not violated.
Solution:
Let the linear combination that maximizes the Uncertainity product be $p\,|+\rangle + q\,|-\rangle$. Since we know that the coefficients are complex in general and that the overall phase is immaterial, we can take p qnd q such that $p = r$ and $q = se^{i\delta}$ where r, s, δ are real numbers.

$$|\alpha\rangle = r\,|+\rangle + se^{i\delta}\,|-\rangle \qquad \leftarrow DC \rightarrow \qquad \langle\alpha| = \langle +|\, r + \langle -|\, se^{-i\delta}$$

Since Operator $S_x \equiv \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)$ and $S_y \equiv \frac{i\hbar}{2}(-|+\rangle\langle -| + |-\rangle\langle +|)$; we can find the expectation value

$$S_x\,|\alpha\rangle = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)(r\,|+\rangle + se^{i\delta}\,|-\rangle) = \frac{\hbar}{2}(se^{i\delta}\,|+\rangle + r\,|-\rangle)$$

$$\begin{aligned}\langle S_x\rangle = \langle\alpha|S_x|\alpha\rangle &= \left[\langle +|\, r + \langle -|\, se^{-i\delta}\right] \frac{\hbar}{2}(se^{i\delta}\,|+\rangle + r\,|-\rangle) \\ &= \frac{\hbar}{2}\left\{rse^{i\delta} + rse^{-i\delta}\right\} \\ &= \frac{\hbar}{2} rs \left\{e^{i\delta} + e^{-i\delta}\right\} \\ &= \frac{\hbar}{2} rs 2\cos(\delta) = \hbar rs \cos\delta\end{aligned}$$

Also we can calculate the expectation value of S_x^2 which is

$$\begin{aligned}\left\langle S_x^2\right\rangle = \langle\alpha|S_x S_x|\alpha\rangle &= \langle\alpha|\, S_x \left(\frac{\hbar}{2}(se^{i\delta}\,|+\rangle + r\,|-\rangle)\right) \\ &= \left[\langle +|\, r + \langle -|\, se^{-i\delta}\right] \frac{\hbar^2}{4}(r\,|+\rangle + se^{i\delta}\,|-\rangle) \\ &= \frac{\hbar^2}{4}(r^2 + s^2) = \frac{\hbar^2}{4} \qquad \text{(By normalization condition)}\end{aligned}$$

Which can be use to calculate the dispersion of S_x as

$$\left\langle (\Delta S_x)^2 \right\rangle = \left\langle S_x^2 \right\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - \hbar^2 r^2 s^2 \cos^2(\delta) = \frac{\hbar^2}{4}\left(1 - 4r^2 s^2 \cos^2(\delta)\right)$$

By similar procedure we can calculate $\left\langle(\Delta S_y)^2\right\rangle = \frac{\hbar^2}{4}(1 - 4r^2s^2\sin^2(\delta)$. So their product is

$$\begin{aligned}
\left\langle(\Delta S_x)^2\right\rangle\left\langle(\Delta S_y)^2\right\rangle &= \frac{\hbar^2}{4}\left(1 - 4r^2s^2\cos^2(\delta)\right)\cdot\frac{\hbar^2}{4}\left(1 - 4r^2s^2\sin^2(\delta)\right) \\
&= \frac{\hbar^4}{16}(1 - 4r^2s^2\sin^2(\delta) - 4r^2s^2\cos^2(\delta) + 16r^4s^4\sin^2(\delta)\cos^2(\delta)) \\
&= \frac{\hbar^2}{16}(1 - 4r^2s^2 + 16r^4s^44\sin^2(\delta)\cos^2(\delta)) \\
&= \frac{\hbar^2}{16}(1 - 4r^2s^2 + 4r^4s^4\sin^2(2\delta))
\end{aligned}$$

Since r and s are constrained by normalization as $s = \sqrt{1-r^2}$. The two parameters for the variation of the product is δ and r (or s). The since $\sin^2(2\delta)$ can attain the maximum value of 1 whhich gives $\sin^2(2\delta) = 1; \Rightarrow 2\delta = \frac{\pi}{2} \Rightarrow \delta = \frac{\pi}{4}$. So the uncertainity product reduces to

$$\begin{aligned}
\left\langle(\Delta S_x)^2\right\rangle\left\langle(\Delta S_y)^2\right\rangle &= \frac{\hbar^2}{16}(1 - 4r^2s^2 + 4r^4s^4) \\
&= \frac{\hbar^2}{16}\left(1 - 2r^2s^2\right)^2
\end{aligned}$$

The maximum value of this expression occurs when $2r^2s^2$ is the minimum, which by inspection is 0 at $r = 0$. Using this value $r = 0$ in normalization condition $r^2 + s^2 = 1$ gives $s = \pm 1$. So the linear combination we started reduces to

$$|\alpha\rangle = 0\,|+\rangle \pm e^{i\frac{\pi}{4}}\,|-\rangle = \left(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right)\Big|-\Big\rangle$$

■

4. Show that either $[A, B] = 0$ or $[B, C] = 0$ is sufficient for $\langle c'|a'\rangle$ to be
Solution:
Let the common eigenket of compatible operators A, B be $|a', b'\rangle$. Since they are observable the set of these eigenkets form a complete set let them be $|a', b'\rangle, |a'', b''\rangle \cdots |a^n, b^n\rangle$ for n state (dimensional) system. In the first way of invividually measuring the outcomes of B observables the total probability of observing $\left|c^1\right\rangle$ state is then

$$\left|\langle c^1|a^1\rangle\right|^2 = \sum_i \left|\langle c^i|a^i, b^i\rangle\right|^2 \left|\langle a^i, b^i|s\rangle\right|^2$$

■