PHYS :516 Quantum Mechanics I

Homework #1

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1. (a) Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle, \langle a''|\alpha\rangle, \cdots$ and $\langle a'|\beta\rangle, \langle a''|\beta\rangle, \cdots$ are all known, where $|a'\rangle, |a''\rangle, \cdots$ form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in this basis.

Solution:

We know every ket can be written as the sum of its component in the 'direction' of base ket (completeness) so $|\gamma\rangle$ can be written as

$$|\gamma\rangle = \sum_i \left|a^i\right\rangle \left\langle a^i\middle|\gamma\right\rangle$$

Let the operator $|\alpha\rangle\langle\beta|$ act on an arbitrary ket $|\gamma\rangle$.

$$|\alpha\rangle\langle\beta|\gamma\rangle = \sum_i |\alpha\rangle \left\langle\beta\middle|a^i\right\rangle \left\langle a^i\middle|\gamma\right\rangle$$

So the component of this $|\alpha\rangle\langle\beta|\gamma\rangle$ in the direction of another eigen ket $\left|a^j\right\rangle$ is then given by the inner product of it with $\left|a^j\right\rangle$

$$\left(|\alpha\rangle\langle\beta|\gamma\rangle\right)_j = \left\langle a^j\middle|\alpha\right\rangle\langle\beta|\gamma\rangle = \sum_i \underbrace{\left\langle a^j\middle|\alpha\right\rangle\left\langle\beta\middle|a^i\right\rangle}_{N\times N}\left\langle a^i\middle|\gamma\right\rangle \tag{1}$$

This anove expression can be written as the matrix form as

$$\begin{bmatrix} (|\alpha\rangle\langle\beta|\gamma\rangle)_1 \\ (|\alpha\rangle\langle\beta|\gamma\rangle)_2 \\ \vdots \\ (|\alpha\rangle\langle\beta|\gamma\rangle)_N \end{bmatrix} = \begin{bmatrix} \left\langle a^1\middle|\alpha\right\rangle\left\langle\beta\middle|a^1\right\rangle & \left\langle a^1\middle|\alpha\right\rangle\left\langle\beta\middle|a^2\right\rangle & \cdots & \left\langle a^1\middle|\alpha\right\rangle\left\langle\beta\middle|a^N\right\rangle \\ \left\langle a^2\middle|\alpha\right\rangle\left\langle\beta\middle|a^1\right\rangle & \left\langle a^2\middle|\alpha\right\rangle\left\langle\beta\middle|a^2\right\rangle & \cdots & \left\langle a^2\middle|\alpha\right\rangle\left\langle\beta\middle|a^N\right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle a^N\middle|\alpha\right\rangle\left\langle\beta\middle|a^1\right\rangle & \left\langle a^N\middle|\alpha\right\rangle\left\langle\beta\middle|a^2\right\rangle & \cdots & \left\langle a^N\middle|\alpha\right\rangle\left\langle\beta\middle|a^N\right\rangle \end{bmatrix} \begin{bmatrix} (|\gamma\rangle)_1 \\ (|\gamma\rangle)_2 \\ \vdots \\ (|\gamma\rangle)_N \end{bmatrix}$$

Since every $\langle a'|\beta\rangle$ is known each element $\left\langle\beta\middle|a^i\right\rangle$ in above matrix can be written as the complex conjugate of known $\left\langle a^i\middle|\beta\right\rangle^*$. So the matrix representation becomes

$$|\alpha\rangle\langle\beta| \equiv \begin{bmatrix} \left\langle a^1\middle|\alpha\right\rangle\left\langle a^1\middle|\beta\right\rangle^* & \left\langle a^1\middle|\alpha\right\rangle\left\langle a^2\middle|\beta\right\rangle^* & \cdots & \left\langle a^1\middle|\alpha\right\rangle\left\langle a^N\middle|\beta\right\rangle^* \\ \left\langle a^2\middle|\alpha\right\rangle\left\langle a^1\middle|\beta\right\rangle^* & \left\langle a^2\middle|\alpha\right\rangle\left\langle a^2\middle|\beta\right\rangle^* & \cdots & \left\langle a^2\middle|\alpha\right\rangle\left\langle a^N\middle|\beta\right\rangle^* \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle a^N\middle|\alpha\right\rangle\left\langle a^1\middle|\beta\right\rangle^* & \left\langle a^N\middle|\alpha\right\rangle\left\langle a^2\middle|\beta\right\rangle^* & \cdots & \left\langle a^N\middle|\alpha\right\rangle\left\langle a^N\middle|\beta\right\rangle^* \end{bmatrix}$$

Which is the required matrix representation of $|\alpha\rangle\langle\beta|$ ■

(b) Consider of spin $\frac{1}{2}$ system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|S_z = \hbar/2\rangle$ and $|S_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (S_z diagonal) basis.

Solution:

The basis kets are $|S_z;+\rangle \equiv |+\rangle$ and $|S_z;-\rangle \equiv |-\rangle$. The state ket $|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$. So the four matrix elements are

$$\langle\alpha|+\rangle = 1; \qquad \langle\alpha|-\rangle = 0$$
$$\langle\beta|+\rangle = \frac{1}{\sqrt{2}}(1+0) = \frac{1}{\sqrt{2}} \qquad \langle\beta|-\rangle = \frac{1}{\sqrt{2}}(0+1) = \frac{1}{\sqrt{2}}$$

The required matrix representation is

$$\begin{bmatrix} \langle+|\alpha\rangle\langle\beta|+\rangle & \langle+|\alpha\rangle\langle\beta|-\rangle \\ \langle-|\alpha\rangle\langle\beta|+\rangle & \langle-|\alpha\rangle\langle\beta|-\rangle \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Which is the required matrix representation of the operator in the basis $|S_z;+\rangle$ and $|S_z;-\rangle$ ■

2. Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k, \qquad \{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij},$$

Where, $S_x = \frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|)$, $\quad S_y = \frac{i\hbar}{2}(-|+\rangle\langle-| + |-\rangle\langle+|)$, $\quad S_z = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$

Solution:

$$S_xS_y = \frac{i\hbar^2}{4}\{-|+\rangle\langle-|+\rangle\langle-| + |+\rangle\langle-|-\rangle\langle+| - |-\rangle\langle+|+\rangle\langle-| + |-\rangle\langle+|-\rangle\langle+|\} = \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\}$$
$$S_yS_x = \frac{i\hbar^2}{4}\{|+\rangle\langle-|+\rangle\langle-| - |+\rangle\langle-|-\rangle\langle+| + |-\rangle\langle+|+\rangle\langle-| - |-\rangle\langle+|-\rangle\langle+|\} = -\frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\}$$
$$[S_x, S_y] = S_xSy - S_yS_x = \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\} + \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\} = \frac{i\hbar^2}{2}\{|+\rangle\langle-| - |-\rangle\langle-|\} = i\hbar S_z$$

Since $[S_x, S_y] = i\hbar S_z$ it immediately follows that $[S_y, S_x] = -i\hbar S_z$ because $[A, B] = -[A, B]$. Collecting all these leads to $[S_i, S_j] = i\varepsilon_{ijk}S_k$.

$$\{S_x, S_y\} = S_xS_y + S_yS_x = \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\} - \frac{i\hbar^2}{4}\{|+\rangle\langle-| - |-\rangle\langle-|\} = 0$$
$$\{S_x, S_x\} = S_xS_x + S_xS_x = 2S_xS_x = 2\frac{\hbar^2}{4}\left\{\underbrace{\langle+|+\rangle + \langle-|-\rangle}_{\text{Identity operator}}\right\} = \frac{\hbar^2}{2}$$

Similarly $\{S_x, S_x\} = \frac{\hbar^2}{2}; \{S_y, S_y\} = \frac{\hbar^2}{2}; \{S_z, S_z\} = \frac{\hbar^2}{2}; \{S_x, S_y\} = 0; \{S_y, S_z\} = 0; \{S_z, S_x\} = 0;$ which can be copactly written as $\{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij}$ for each operator leads to the required relation of the commutation and anti commutation relation of the given operators. ■

3. The hamiltonian operator for a two-state system is given by

$$h = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|),$$

where a is a number with the dimension of energy. find the energy eigenvalues and the corresponding energy eigenkets (as a linear combinations of $|1\rangle$ and $|2\rangle$)

Solution:

let the energy eigenket be $|\alpha\rangle = p|1\rangle + q|2\rangle$. let the eigenvalue of this energy eigenket be a'. operating this eigenket by the given hamiltonian operator we get.

$$\begin{aligned} h|\alpha\rangle &= a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)(p|1\rangle + q|2\rangle) \\ &= a(p|1\rangle + p|2\rangle - q|2\rangle + q|1\rangle) \\ &= a[(p+q)|1\rangle + (p-q)|2\rangle] \end{aligned}$$

since we assume a' is the eigenvaue of this ket we must have $h|\alpha\rangle = a'|\alpha\rangle$ thus

$$a'(p|1\rangle + q|2\rangle) = a[(p+q)|1\rangle + (p-q)|2\rangle]$$

since $|1\rangle$ and $|2\rangle$ are independent kets, the coefficient of each ket on lhs and rhs must equal. comparing the coefficients we have

$$\begin{aligned} -ap + (a+a')q = 0; &\qquad \rightarrow p = \frac{a+a'}{a}q \\ (a-a')p + aq = 0; &\qquad \rightarrow (a-a')\frac{a+a'}{a}q + aq = 0; \qquad \rightarrow a^2 - a'^2 + a^2 = 0; \quad a' = \pm\sqrt{2}a \end{aligned}$$

so the required eigenvalues of the operator are $\pm\sqrt{2}a$.

the coefficient

$$p = \frac{a \pm \sqrt{2}a}{a}q = (1 \pm \sqrt{2})q$$

. since we have a free choice of one of the parameters we choose p and q such that the energy eigenket is normalized. so the required eigenket is

$$|\alpha\rangle = \frac{1}{\sqrt{1 + (1 \pm \sqrt{2})^2}}((1 \pm \sqrt{2})|1\rangle + |2\rangle) = \frac{1}{\sqrt{4 \pm 2\sqrt{2}}}\Big((1 \pm \sqrt{2})|1\rangle + |2\rangle\Big)$$

the above expression $|\alpha\rangle$ gives the energy eigenket corresponding to eigenvalue $\pm\sqrt{2}a$. ■

4. A beam of spin $\frac{1}{2}$ atom goes through a series of stern-gerlach-type measurements as follows:

(a) the first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.

(b) the second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $s \cdot \hat{n}$ with $\hat{n}$ making an angle β in the $xz-$plane wi respect to the z-axis.

(c) the third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

what is the intensity of the final $s_z = -\hbar/2$ bean when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? how must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?

Solution:
The First Stern-Gerlach measurement in S_z is independent of the second Stern-Gerlach measurement in $\hat{n}$ the probability of atom passing through each component is $\frac{1}{2}$. Due to this measurement and the $S_n = -\hbar/2$ being rejected the system essentially forgets the previous measurement and the atom still come out 50%. So the fraction of atoms passing through the third SG apparatus in S_z direction is still $\frac{1}{2}$. So the total fraction of atoms passing thourhg the third SG apparatus is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = 25\%$,

If the second SG apparatus is oriented parallel to the first apparatus then it essentially measures the $|S_z;+\rangle$ state of the atom which was what came from the first apparatus so it lets 100% of the atom in $|S_z;+\rangle$ state. And the third apparatus will let half of the second which is 50% of the atoms which passed through the first apparatus. Orienting the second SG apparatus parallel to the first will let all of the atoms, this is the required condition of maximizing the output of third. ■

5. Prove that if operator $X = |\beta\rangle\langle\alpha|$, then the hermitian conjugate of the operator is $X^\dagger = |\alpha\rangle\langle\beta|$.
Solution:
Acting this operator $X = |\beta\rangle\langle\alpha|$ on an arbitrary ket $|\gamma\rangle$

$$\begin{aligned}
& X|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle \\
\Rightarrow\quad & \langle\gamma| X^\dagger = \langle\beta| \langle\alpha|\gamma\rangle^* && (\because \text{Dual correspondence}) \\
\Rightarrow\quad & \langle\gamma| X^\dagger = \langle\beta| \langle\gamma|\alpha\rangle && (\because \langle\gamma|\alpha\rangle = \langle\alpha|\gamma\rangle^*) \\
\Rightarrow\quad & \langle\gamma| X^\dagger = \langle\gamma|\alpha\rangle \langle\beta| && (\because c|\delta\rangle = |\delta\rangle c) \\
\Rightarrow\quad & \langle\gamma| X^\dagger = \langle\gamma| (|\alpha\rangle\langle\beta|) && (\because \text{ Associative property}) \\
\Rightarrow\quad & X^\dagger = |\alpha\rangle\langle\beta|
\end{aligned}$$

Thus if $X = |\beta\rangle\langle\alpha|$ then $X^\dagger = |\alpha\rangle\langle\beta|$ is shown as required. ■