PHYS 512: Electromagnetic Theory II

Homework #5

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1. **(Jackson 9.3)** Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulation gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V \cos \omega t$. In the long wavelength limit, find the radiation fields, the angular distribution of radiated power and the total radiated power from the sphere.
Solution:
Two opposite charged halves of sphere creates a dipole so the dipole term in the potential expansion is the dominant term. The dominant term on the potential expansion in terms of Legendre polynomial expansion is

$$\Phi = V \frac{3}{2} \frac{R^2}{r^2} \cos\theta$$

The potential due to electric dipole pointing iz the positive z direction is given by

$$\Phi_{\text{dip}} \frac{1}{4\pi\epsilon_0} \frac{p}{r^2} \cos\theta$$

The donimant term must be equal to the dipole potential. Equating these

$$V \frac{3}{2} \frac{R^2}{r^2} \cos\theta = \frac{1}{4\pi\epsilon_0} \frac{p}{r^2} \cos\theta$$
$$\implies \vec{p} = 6\pi\epsilon_0 V R^2 \hat{\boldsymbol{z}}$$

The potential in the sphere is oscillation with the frequency ω as $\cos \omega t$, The magnetic field of such oscillating field can be written as

$$\vec{B} = \frac{\mu_0 c k^2 p}{4\pi} \left(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{p}} \right) \frac{e^{i(kr - \omega t)}}{r}$$

Subsisting the value of the dipole moment we get

$$\vec{B} = -\frac{3}{2} \frac{V k^2 R^2}{c} \frac{e^{i(kr-\omega t)}}{r} \sin\theta \hat{\boldsymbol{\phi}}$$

The electric field is similarly given by

$$\vec{E} = -\frac{k^2 p}{4\pi\epsilon_0} \hat{\boldsymbol{k}} \times \left(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{p}} \right) \frac{e^{i(kr-\omega t)}}{r}$$

Simplifying the vector cross products we simplify this down to

$$\vec{E} = -\frac{3}{2} V k^2 R^2 \frac{e^{i(kr-\omega t)}}{r} \sin\theta \hat{\boldsymbol{\theta}}$$

Now the overall radiated power per solid angle is given by

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{2} \mathrm{Re} \left(r^2 \hat{\boldsymbol{r}} \cdot \vec{E} \times \vec{H}^* \right)$$

Subsisting the values of electric field and magnetic field in this expression we get

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{2}\mathrm{Re}\left(r^2\hat{\boldsymbol{r}} \cdot \left\{-\frac{3}{2}Vk^2R^2\frac{e^{i(kr-\omega t)}}{r}\sin\theta\hat{\boldsymbol{\theta}}\right\} \times \left\{-\frac{1}{\mu_0}\frac{3}{2}\frac{Vk^2R^2}{2}\frac{e^{i(kr-\omega t)}}{r}\sin\theta\hat{\boldsymbol{\phi}}\right\}\right)$$

Since $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}}$, the above expression simplifies to

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{9}{8}\frac{V^2k^4R^4}{\mu_0 c}\sin^2\theta$$

The total radiated power is thus the integral of the above expression over the total solid angle in the entire spherical shell

$$P = \oint \frac{\mathrm{d}P}{\mathrm{d}\Omega}d\Omega = \oint \frac{9}{8}\frac{V^2k^4R^4}{\mu_0 c}\sin^2\theta d\Omega$$

Integral of of the quantity $\sin^2\theta$ over the total solid angle is just $\frac{8}{3}$ thus giving us the final expression

$$P = \frac{3\pi V^2K^4R^4}{\mu_0 c}$$

This gives the total radiated power. □