PHYS 512: Electromagnetic Theory II

Homework #4

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1. **(Jackson 8.4)** Transverse electric and magnetic waves are propagated along a hollow, right circular cylinder with inner radius R and conductivity σ

   (a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in term of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part assume that the conductivity of the cylinder is infinite.
   Solution:
   The eivenvalue equation for both the TE and TM mode is

$$\left(\nabla_t^2 + \gamma^2\right)\psi(r,\phi) = 0$$

where $\psi(R,\phi) = 0$. For TE mode there is no axial electric field, so we can solve for B_z. There are no charges and currents in the waveguide so they obety the homogenous wave equation

$$\nabla^2 B_z - \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2} = 0$$

The wave is free along the axis of waveguide. so we can ssume that the solution for the magnetic field has harmonic dependence in time in the direction of propogation thus we can write

$$B = B_z e^{ikz - i\omega t}$$

Substuting this in the expression for the homognneous equation we get

$$\nabla_t^2 B_z = \left(\frac{\omega^2}{c^2} - k^2\right) B_z$$

The laplacian operator in this expression is only in the transerse directionn. Because of the cylindrical symmetri we can write the lapalacian in the cylindrical coordinate system as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 B_z}{\partial \phi^2} = \left(k^2 - \frac{\omega^2}{c^2}\right) B_z$$

Making a substution

$$k' = \left(\frac{\omega^2}{c^2} - k^2\right)$$

If we assume the solution of the magnetic field $B_z = (k'r)e^{im\phi}$ we get

$$r^2\frac{\partial^2 (k'r)}{\partial r^2} + r\frac{\partial R(k'r)}{\partial r} + \left(k^2 r^2 - m^2\right) R(kr) = 0$$

This differential equation can be converted to a bessel differential equation with $kr = x$. The equation then becomes

$$x^2\frac{\partial^2 R(x)}{\partial x^2} + x\frac{\partial R(x)}{\partial x} + \left(x^2 - ^2\right) R(x) = 0$$

This is besseldifferential equatio. the solution of this equation is

$$R(x) = AJ_m(x) + BN_m(x)$$

Substuting this in the magnetic field expression we get

$$B_z = \left(A_m(x) + BN_m(x)\right) e^{ikz-\omega t+m\phi}$$

Since the functions $N_m(x)$ blow up at $x = 0$, and that the field is finite at the axis we have to have $B = 0$. The solution then becomes

$$B_z = AJ_m(k'r)e^{i(kz-\omega t+m\phi)}$$

At the surface of the perfect conductior constuting the wallsof eh waveguide, we have theboundary condidtion

$$\left.\frac{\partial B_z}{\partial r}\right|_{r=R} = 1$$

Applying this condition we get

$$\left(\frac{\partial}{\partial r}\left(J_m(k'r)\right)\right)_{r=R} = 0$$

The zeros of the equation are simply the zeros of derivatives of bessel functions. Assuming the zeros are α_{mn} we get

$$k'r = \alpha_{mn} \implies k' = \frac{\alpha_{mn}}{R}$$

Substuting this for the expression relating k and k' we get

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{\alpha_{mn}^2}{R^2}}$$

Thus the magnetic field becomes

$$B_z = AJ_m(\alpha_{mn}\frac{r}{R})e^{i(kz-\omega t+m\phi)}$$

For TE mode the axial electric field obeys same equation and we get similar differential equation whose solution is

$$E_z = AJ_m(k'r)e^{i(kz-\omega t+m\phi)}$$

The boundary condition is tha the electric field is zero at the walls $E_z(r = R) = 0$ so we get Now instead of the zeros of derivatives of bessel function the zeros are at the zeros of bessel function β_{mn} so we get

$$k' = \frac{\beta_{mn}}{R}$$

The solution then becomes

$$E_z = AJ_m(\beta_{mn}\frac{r}{R})e^{i(kz-\omega t+m\phi)}$$

The cutoff frequencies are the frequencies where the wavenumber equals zero. So we get

$$\omega_{mn} = c\frac{\beta_{mn}}{R}\text{for TE mode}$$
$$\omega_{mn} = c\frac{\alpha_{mn}}{R}\text{for TM mode}$$

These are the required cutoff frequencies for TE mode and TM mode. □

(b) Calculate for TM mode the attenuation constants of the waveguide as a function of frequency for the lowest two distinct modes and plot them as a function of frequency.

2. **(Jackson 8.6)** A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with the flat end faces. Determine the resonant frequencies of the cavity for all types of waves. With $\frac{1}{\sqrt{\mu}R}$ as a unit of frequency. Plot the lowest resonant frequencies of each type as s function of $\frac{R}{L}$ for $0 < \frac{R}{L} < 2$. Does the same mode have the lowest frequency for $\frac{R}{L}$?
Solution:
For the cavity, the normal modes in TM modes are given by

$$\psi(,\phi) = E_{Jm}(\gamma_{mn} r)e^{\pm im\phi} \qquad \text{where} \gamma_{mn} = \frac{x_{mn}}{R}$$

Here x_{mn} are the zeros of bessle function J_m. As given in Jackson eq. 8.81 we get the resonant frequency

$$w_{mnp} = \frac{1}{\sqrt{\mu\epsilon}R}\sqrt{x_{mn}^2 + \left(\frac{p\pi R}{L}\right)^2}$$

The zeros of bessel are

$$x_{01} = 2.405, x_{12} = 3.832, x_{21} = 5.136 and$$

□