PHYS 512: Electromagnetic Theory II

Homework #3

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May 16, 2019

1. **(Jackson 7.12)** The time dependence of electrical distrubaances in good conductors is governed by the frequency-dependent conductivity. Consider longitudinal electric fields in a conductor, using Ohm's law, the contunuity equation, and the different form of Coulomb's law.

   (a) Show that the time-Fourier transformed charge density satisfies the equation

$$[\sigma(\omega) - i\omega\epsilon_0]\,\rho(\boldsymbol{x}, \omega) = 0$$

Solution:
Let us assume the time varying quantities be charge density $\rho(t)$, current density $J(t)$ and electric field $E(t)$. Taking the fourier transform to take to frequency space

$$\rho(\omega) = \frac{1}{\sqrt{2\pi}} \int \rho(t) e^{i\omega t} dt$$
$$\boldsymbol{J}(\omega) = \frac{1}{\sqrt{2\pi}} \int \boldsymbol{J}(t) e^{i\omega t} dt$$
$$\boldsymbol{E}(\omega) = \frac{1}{\sqrt{2\pi}} \int \boldsymbol{E}(t) e^{i\omega t} dt$$

Now The contuinity equation

$$\nabla J = -\frac{\partial \rho}{\partial t}$$

In the frequency space, this becomes

$$\nabla \cdot \boldsymbol{J}() = i\omega\rho(\omega)$$

The Ohm's law relates chage current density and electric field as,

$$\boldsymbol{J}(\omega) = \sigma(\omega)\boldsymbol{E}(\omega)$$

The coloumbs law can be used express the relation between the electric field and charge density as

$$\nabla \cdot \boldsymbol{E}(\omega) = \frac{\rho(\omega)}{\epsilon_0}$$

Combining all these we obtain

$$(\sigma(\omega) - i\omega\epsilon_0)\,\rho(\omega) = 0$$

This is the required expression. □

   (b) Using the representation $\sigma(\omega) = \sigma_0/(1 - i\omega\tau)$ where $\sigma_0 = \epsilon_0\omega_p^2\tau$ and τ is a dampinng time, show that the approximation $\omega_p\tau \gg 1$ any initial distrubance will oscillate with plasma frqueynyc and decay amplitude with a decay constatn $\lambda = 1/2\tau$.

Solution:
Using the representation $\sigma(\omega) = \sigma_0(1 - i\omega\tau)$ we get

$$\left(\frac{\sigma_0}{1 - i\omega\tau} - i\omega\epsilon_0\right)\rho(\omega) = 0$$

substuting $\sigma_0 = \epsilon_0\omega_p^2\tau$

$$\left[\frac{\omega_p^2\tau}{1 - i\omega\tau} - i\omega\right] = 0$$

this is a quadratic equation in ω which can be rearranged to get $\tau\omega^2 + i\omega - \omega_p^2\tau = 0$. The solutions are

$$\omega = \frac{-i \pm \sqrt{4\tau^2\omega_p^2 - 1}}{2\tau}$$

Using the given approximation $\omega_p\tau \gg 1$ we obtain

$$\omega = \pm\omega_p - \frac{i}{2\tau}$$

This shows that in frequency space the signal is delayed by $\frac{1}{2\tau}$. Reverting back to time space with inverse fouerir transform we get

$$f(t) = \mathcal{F}^{-1}F(\omega_p - i\frac{1}{2\tau})$$
$$f(t) = f_0(t)e^{-t/2\tau}$$

This shows that the signal is decays at the rate $\frac{1}{2\tau}$ □

2. **(Jackson 7.19)** An approximately monochormatic plane wave packet in one dimension has the instantaneous form $u(x, 0) = f(x)e^{k_0 x}$, with $f(x)$ the modulation envelope. For each of teh forms $f(x)$ below, calculate the wave number spectrum $|A(k)|^2$ of the packet, sketch $|u(x, 0)|^2$ and $|A(k)|^2$, evaluate explicitly the rms deviations from the means Δx and Δk

   (a) $f(x) = Ne^{\alpha|x|/2}$
   Solution:
   The initial waveform for this problem is $u(x, 0) = Ne^{k_0 x = \alpha|x|/2}$ The wave number spectrum can be obtained as

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{ikx}ds$$
$$= \frac{1}{\sqrt{2\pi}}N\int_{-\infty}^{\infty} e^{ikx + ik_0 x - \alpha\frac{|x|}{2}}dx$$

This integral is a function of α and since it is even function of x we can write above integral as

$$A(k) = \frac{1}{\sqrt{2\pi}}2N\left[\int_0^{\infty} \cos(k - k_0)xe^{-\alpha x/2}\right]$$

This integral can be computed and the final expression fo rthe integral gives

$$A(k) = \frac{1}{\sqrt{2\pi}}\left[\frac{N\alpha}{\alpha^2/4 + (k - k_0)^2}\right]$$

The mean square value for a function $f(x)$ is given by the expression

$$\text{Mean Square } = \frac{\int\limits_{-\infty}^{\infty} x^2 \left[f(x)\right]^2 dx}{\int\limits_{-\infty}^{\infty} \left[f(x)\right]^2 dx}$$

For the mean squred deviation of x we can write

$$\sigma_x^2 = \frac{\int\limits_{-\infty}^{\infty} x^2 e^{-\alpha|x|} dx}{\int\limits_{-\infty}^{\infty} e^{\alpha|x|} dx}$$

These integrals can be calulated with gamma functions, and the final result after integration is

$$\Delta x = \frac{\sqrt{2}}{\alpha}$$

Similarly with same token for the spread of $A(x)$ we obtain

$$\Delta k = \sqrt{\frac{\int\limits_{-\infty}^{\infty} k^2 \left[\frac{1}{\alpha^2/4 + k^2}\right]^2 dk}{\int\limits_{-\infty}^{\infty} \left[\frac{1}{\alpha^2/4 + k^2}\right]^2 dk}}$$

This integral was obtained using computer algebra system and the final expression is

$$\Delta k = \frac{\alpha}{2}$$

Checking for the product of $\Delta x \Delta k$ we get

$$\Delta x \Delta k = \sqrt{2}/2 = \frac{1}{\sqrt{2}} \geq \frac{1}{2}$$

□

(b) $f(x) = Ne^{\alpha^2 x^2}/4$
Solution:
Taking the fouerie transform to get the frequency component functions

$$A(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x,0) e^{ikx} dx$$

This can be integrated for the geven initial shape as

$$A(k) = \frac{1}{\sqrt{2\pi}} N \int\limits_{-\infty}^{\infty} e^{ik_0 x - ikx - \alpha^2 x^2/4}$$

Thsi can be calculate to obtain

$$A(k) = N\sqrt{2/\alpha^2} e^{-(k-K_0)^2/\alpha^2}$$

The spread can bew be similarly calculaged as above

$$\Delta x = \sqrt{\frac{\int\limits_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2/2} dx}{\int\limits_{-\infty}^{\infty} e^{-\alpha^2 x^2/2} dx}}$$

The integrals can be calculated using gamm function identities and the final expression (with computer algebra system used) is

$$\Delta x = \frac{1}{\alpha}$$

Similarly the spread in the frequency component can be calculated

$$\Delta k = \sqrt{\frac{\int\limits_{-\infty}^{\infty} k^2 e^{-2k^2/\alpha^2} dk}{\int\limits_{-\infty}^{\infty} e^{-2k^2/\alpha^2} dk}}$$

This was also solved using computer algebra system to obtain

$$\Delta k = \frac{\alpha}{2}$$

For this signal also the inequality $\Delta x \Delta k \geq \frac{1}{2}$ holds as

$$\Delta x \Delta k = \frac{1}{\alpha} \cdot \frac{\alpha}{2} = \frac{1}{2} \geq \frac{1}{2}$$

So both the wave train satisfy the uncertainity principle. □

3. **(Jackson 8.2)** A transmission line sonsisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ, as shown, is filled with a uniform lossless dielectric (μ, ϵ). A TEM mode is propagated along this line,

   (a) Show that the time-averaged power flow along the line is

   $$P = \sqrt{\frac{\mu}{\epsilon}} \pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$$

   where H_0 is the peak value of the azimuthal magnetic field and the surface of the innser conductor.
   Solution:
   By definition a TEM mode is a signle-frequency wave cmponent with both the electric field and magnetic field transverse to the directio of propagation along the waveaxis. The innder conductor has to have some charge per unit length, say λ. With a cylindrical gaussian surface around the inner conductor we find the electric field is

   $$\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon\rho}\hat{\boldsymbol{\rho}}$$

   Since the waveguide axis is along the z axis, the magnetic field can be obtained from electric field as

   $$\boldsymbol{B} = \sqrt{\mu\epsilon}\hat{\boldsymbol{z}} \times \boldsymbol{E} = \sqrt{\mu\epsilon}\frac{\lambda}{2\pi\epsilon\rho}\hat{\boldsymbol{\phi}}$$

Since given in the problem that H_0 is the peak value of magnetic field in the inner conductor, we obtain H_0 as

$$H_0 = \boldsymbol{H}(\rho = a) = \boldsymbol{B}(\rho = a)\frac{1}{\mu\epsilon} = \frac{1}{\sqrt{\mu\epsilon}}\frac{\lambda}{2\pi a}$$

This expression gives the total charge per unit length equal to

$$\lambda = 2\pi a H_0 \sqrt{\mu\epsilon}$$

Substuting this in the expression for electric field we get

$$\boldsymbol{E} = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{q}{\rho}\hat{\boldsymbol{\rho}} \qquad \boldsymbol{B} = \mu H_0 \frac{a}{\rho}\hat{\boldsymbol{\phi}}$$

These two fields are correct for the static problem. Introducing the time dependence in the waveguide we obtain

$$\boldsymbol{E} = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{q}{\rho} e^{ikz - i\omega t}\hat{\boldsymbol{\rho}} \qquad \boldsymbol{B} = \mu H_0 \frac{a}{\rho} e^{ikz - i\omega t}\hat{\boldsymbol{\phi}}$$

Now we can calculate the energy flux using the poynting vector as

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$
$$\boldsymbol{S} = \frac{1}{\mu}\left[\sqrt{\mu/\epsilon} H_0 \frac{a}{\rho} e^{ikz - i\omega t}\right] \times \left[\mu H_0 \frac{a}{\rho} e^{ikz - i\omega t}\right]$$

Since in general we the quantity H_0 is a complex number we can write this as

$$H_0 = |H_0| e^{i\theta}$$

Substuting this in aboe expression and carrying out the corss product we get

$$\boldsymbol{S} = \sqrt{\frac{\mu}{\epsilon}} |H_0|^2 \frac{a^2}{\rho^2} \cos^2\left(kz - \omega t + \theta\right)\hat{\boldsymbol{z}}$$

The time averaged power flux is thus the average of above expression. But the average of $\cos^2$ is

$$\left\langle \cos^2 \alpha \right\rangle = \frac{1}{2}$$

So we get

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} |H_0|^2 \frac{a^2}{z^2}\hat{\boldsymbol{z}}$$

The total power can now be obtained by integrating the power flux over the whole area

$$P = \oint_S \hat{\boldsymbol{z}} \cdot \langle \boldsymbol{S} \rangle \, dA$$
$$= \int_0^{2\pi}\int_a^b \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} |H_0|^2 \frac{a^2}{\rho^2} \rho d\rho d\phi$$

The integral in ϕ is just the value 2π and the rho integral is just logarithm. So we get

$$P = \sqrt{\frac{\mu}{\epsilon}} \pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right) \tag{1}$$

This is the requried power flow. □

(b) Show that hte transmitted power is attenuated along the line as

$$P(z) = P_0 e^{-2\gamma z}$$

where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\frac{1}{b}+\frac{1}{b}}{\ln\left(\frac{b}{a}\right)}$$

Solution:
The rate of power loss per unit area with skin depth δ is given by

$$\frac{\mathrm{d}P}{\mathrm{d}a} = \frac{1}{4}\mu_c\omega\delta\left|\boldsymbol{H}_{\parallel}\right|^2$$

The area element in can be written as

$$da = \rho d\phi dz$$

Using this expression in the power flow equation we get

$$\frac{\mathrm{d}P}{\mathrm{d}a}\rho d\phi = \frac{1}{4}\mu_c\omega\delta\int_0^{2\pi}\left|\boldsymbol{H}_{\parallel}\right|^2\rho d\phi$$

There are two boundaries the surface so we get

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{\pi}{2}\mu_c\omega\delta\left[\left|\boldsymbol{H}_{\parallel}(a)\right|^2 a + b\left|\boldsymbol{H}_{\parallel}(b)\right|^2\right]$$

The magnetic field part of the expression can be substutited to get

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{\pi}{\sigma\delta}|H_0|^2 a\left[1+\frac{a}{b}\right] \tag{2}$$

As given in the problem , assuming the power loss along the line as

$$P(z) = P_0 e^{-2\gamma z}$$

Differentiating with respect to z we get

$$\frac{\mathrm{d}P(z)}{\mathrm{d}z} = -2\gamma P \qquad \implies \gamma = \frac{-1}{2P}\frac{\mathrm{d}P}{\mathrm{d}z}$$

Substuting P from (1) and its derivative from (2) we get

$$\gamma = \frac{1}{\sqrt{\frac{\mu}{\epsilon}}\pi a^2|H_0|^2\ln\left(\frac{b}{a}\right)}\left(\frac{\pi}{\sigma\delta}|H_0|^2 a\left[1+\frac{a}{b}\right]\right)$$

Simplification yields

$$\gamma = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

This is the required expression. □