PHYS 512: Electromagnetic Theory II

Homework #2

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1. **(Jackson 6.11)** A transverse plane wave is incident normally in vacuum on a perfectly absorbing flat screen

   (a) From a law of conservation of linear momentum , show that the pressuer exerted on teh screen is equal to the field energy per unit volume in the wave.
   Solution:
   We can choose our coordinate system such that the z axis lies along the direction that the plane wave travels. Since electric amd magnetic fields are perpndicular to each other and to the direction of propagation the electric field and magnetic field become

$$\boldsymbol{E} = E\hat{\boldsymbol{\imath}} \qquad \boldsymbol{H} = H\hat{\boldsymbol{\jmath}}$$

The momentum conservation equation for j^{th} component of the momentum is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{P}_{\text{fields}} + \boldsymbol{P}_{\text{mech}}\right)_j = \oint \sum_i T_{ij} n_i da \tag{1}$$

From the way we chose our coordinate system n only has component along the $\hat{\boldsymbol{k}}$ direction, the index for which is 3 so we can replace the summation by a single term

$$\sum_i T_{ij} n_i = T_{3j}$$

The stress energy tensor T_{ij} is given by

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}E^2\delta_{ij}\right) + \mu_0\left(H_iH_j - \frac{1}{2}H^2\delta_{ij}\right)$$

Calculating the the component of the tensor in the rquired direciton we get

$$\begin{aligned} T_{j3} &= \epsilon_0\left(E_3E_j - \frac{1}{2}E^2\delta_{3j}\right) + \mu_0\left(H_3H_j - \frac{1}{2}H^2\delta_{3j}\right) \\ &= \frac{1}{2}\left(\epsilon_0E^2 + \mu_0H^2\right)\delta_{3j} \end{aligned}$$

Again by our choice of coordinate system the component of momentum is also along the z axis so the only non vanishing component of momentum is in that direction.

$$\begin{aligned} P_{j=3} &= \frac{1}{2}\left(\epsilon_0E^2 + \mu_0H^2\right)\delta_{3j}\Big|_{j=3} \\ &= \frac{1}{2}\left(\epsilon_0E^2 + \mu_0H^2\right) \end{aligned}$$

The expression on the right is the expression for the energy density of electromagnetic wave so rthe expression can be written as

$$P_3 = u$$

where u is the energy density. Since the force is the change in momentum per unit time, and since the initial momentum is zero, we get

$$F = (P_3 - 0)/t = \bar{P}_3$$

where $\bar{\cdot}$ is the time averaged momentum. Which is equal to time averaged energy density, thus we get

$$F = \bar{u}$$

This shows that the energy density is energy density of the field. □

(b) In the neighborhood of the earth the flux of electromagnetic energy from the su is pproximately 1.4 kW/m^2. If an interplanetary "sailplane" had a sail of mass $\frac{1g}{m^2}$ of area and negligible other weight, what would be its maximum accleartion in meters persecond squared to the the solare readiation pressure? How does this compare with th eaccleartion due to solare "wind"(corpuscular radiation)?
Solution:
The flux relation to the energy density is $u = \frac{\text{flux}}{c}$ so we get

$$P = \frac{1.4 \times 10^3}{3.0 \times 10^8} = 5 \times 10^{-6}\frac{N}{m^2}$$

So the accleartion (a) can be calculated as

$$a = \frac{PA}{m} = \frac{P}{\frac{m}{A}} = \frac{5 \times 10^{-6}}{1 \times 10^{-3}} = 5 \times 10^{-3}\frac{m}{s^2}$$

The the accleartion is $a = 5 \times 10^-3\frac{m}{s^2}$ □

2. **(Jackson 7.1)** For each set of Stokes parameters set $s_0 = 3, s_1 = -1, s_2 = 2, s_3 = -2$, deduce the amplitude of the electric field, up to an overall phase, in both linear polarization and circular polarization bases and make an accurate drawing similar to Fig. 7.4 showing the lengths of the axes of one of the ellipses and its orientation
Solution:
The stokes parameters are defined for linear polarization with the following relations

$$\begin{aligned} s_0 &= |E_1|^2 + |E_2|^2 \\ s_1 &= |E_1|^2 - |E_2|^2 \\ s_2 &= 2\text{Re}(E_1^* E_2) = 2|E_1||E_2|\cos(\theta_2 - \theta_1) \\ s_3 &= 2\text{Im}(E_1^* E_2) = 2|E_1||E_2|\sin(\theta_2 - \theta_1) \end{aligned}$$

Inverting these relations we get

$$|E_1| = \sqrt{\frac{s_0 + s_1}{2}} = 2 \qquad |E_2| = \sqrt{\frac{s_0 - s_1}{2}} = \sqrt{2}$$
$$\theta_2 - \theta_1 = \text{acos}\left(\frac{s_2}{\sqrt{s_0^2 - s_1^2}}\right) = \frac{\pi}{4}$$

With these paameters we get the components of electric field as

$$\boldsymbol{E} = \left(|E_1|e^{i\theta_1}, |E_2|e^{i\theta_2}\right) = e^{i\theta_1}\left(|E_1|, |E_2|e^{i\theta_2 - i\theta_1}\right)$$

Sice the phase factor in fron is arbitrary we can ignore it because we can always achieve zero phase factor by rotation of choice of axes. Similarly for the circular polarization case we have the stokes parameters defined as

$$\begin{aligned} s_0 &= |E_+|^2 + |E_-|^2 \\ s_1 &= 2|E_+||E_-|\cos(\theta_- - \theta_+) \\ s_2 &= 2|E_+||E_-|\sin(\theta_- - \theta_+) \end{aligned}$$

Similarly inverting these field amplitudes in terms of parameters give

$$|E_+| = \sqrt{\frac{s_0 + s_3}{2}} = \frac{1}{\sqrt{2}} \qquad |E_-| = \sqrt{\frac{s_0 - s_3}{2}} = \sqrt{\frac{5}{2}}$$
$$\theta_- - \theta_+ = \text{acos}\left(\frac{s_1}{\sqrt{s_0^2 - s_3^2}}\right) = \text{acos}\left(\frac{-3}{\sqrt{5}}\right)$$

Now the field cmponents are

$$\boldsymbol{E} = \left(|E_1|e^{i\theta_1}, |E_2|e^{i\theta_2}\right) = e^{i\theta_1}\left(|E_1|, |E_2|e^{i\theta_2 - i\theta_1}\right)$$

With the parameter for E_1 and E_2 and the phase difference the diagram can be plotted. □

3. **(Jackson 7.3)** Two plane semi-infinite slabs of the same uniform , isotropic, nonpermeable, lossless dielectric with index of refraction n are parallel and separated by an air gap ($n = 1$) with width d . A plane electromagnetic wave of frequency ω is incident on the gap from one of teh slabs with the angle fo incidence i . For linear polarization *both* parallel and perpendicular to the plane of incidence

   (a) Calculate the ratio of power transmitted into the second slab to the inciden power and the ratio of reflected to incident power.
   Solution:
   Let i is the incident angle and r is the angle of refraction by snells law we have

$$n \sin i = \sin r$$

   where n is the refractive index. We can rearrange thi sto get

$$\cos r = \sqrt{1 - \sin^2 r} = \sqrt{1 - n^2 \sin^2 i}$$

   The value of $\cos r$ is purely imaginary when i is greater than critical angle for total internal reflection. To find the transmittedn and replected components in terms of th eincident component we can use the interface matching. In the first interface

$$E_p = E_i + E_r = E_+ + E_-$$
$$H_p = n(E_i - E_r)\cos i = (E_+ - E_-)\cos r$$

   Here we have E_p and H_p are the parallel components of electric and magnetic field.
   In the second interface we have

$$E_+ e^{i\boldsymbol{k}\cdot\boldsymbol{d}} + E_- e^{-i\boldsymbol{k}\cdot\boldsymbol{d}} = E_t$$
$$(E_+ e^{i\boldsymbol{k}\cdot\boldsymbol{d}} - E_- e^{i\boldsymbol{k}\cdot\boldsymbol{d}})\cos r = nE_t \cos i$$

   Solving for E_+ and E_- in terms of E_r and E_i we get

$$E_+ = \frac{1}{2}E_i\left(1 + \frac{n\cos i}{\cos r}\right) + \frac{1}{2}E_r\left(1 - \frac{n\cos i}{\cos r}\right)$$
$$E_- = \frac{1}{3}E_i\left(1 - \frac{n\cos i}{\cos r}\right) + \frac{1}{2}E_r\left(1 + \frac{n\cos i}{\cos r}\right) \tag{2}$$

   Similarly the condition with the second interface can be solved to get

$$E_+ = \frac{1}{2}e^{i\boldsymbol{k}\cdot\boldsymbol{d}}E_t\left(1 + \frac{n\cos i}{\cos r}\right)$$
$$E_- = \frac{1}{2}e^{i\boldsymbol{k}\cdot\boldsymbol{d}}E_t\left(1 - \frac{n\cos i}{\cos r}\right) \tag{3}$$

Let us write $\epsilon = \frac{n \cos i}{\cos r}$ Euation. (2) and (3) can be solved to get

$$\frac{E_t}{E_i} = \frac{4\epsilon}{(1+\epsilon)^2 e^{-i\boldsymbol{k}\cdot\boldsymbol{d}} - (1-\epsilon)^2 e^{i\boldsymbol{k}\cdot\boldsymbol{d}}} \tag{4}$$

$$\frac{E_r}{E_i} = \frac{(1-\epsilon^2)(e^{i\boldsymbol{d}\cdot\boldsymbol{k}})}{(1+\epsilon)^2 e^{i\boldsymbol{k}\cdot\boldsymbol{d}} - (1-\epsilon)^2 e^{i\boldsymbol{k}\cdot\boldsymbol{d}}} \tag{5}$$

This gives the ratio of transmitted to reflected amplitudes. The power is proportional to the square amplitudes so the ratio of transmitted power to the incident power is

$$\frac{P_t}{P_i} = \frac{E_t^2}{E_i^2} = \left[\frac{E_t}{E_i}\right]^2$$

and similarly the reflected power ratio is

$$\frac{P_r}{P_i} = \frac{E_r^2}{E_i^2} = \left[\frac{E_r}{E_i}\right]^2$$

These are the required ratios where the ratios of amplitudes are calculated. □

(b) for i greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d in units of wavelength in the gap.
Solution:
In the equations (4) and (5) we can write the ration ϵ and the phase $\boldsymbol{k}\cdot\boldsymbol{d}$ as purely imaginary numbers and simplify those quations to get the function of th ratios. So assuming the phase and the ratio to be complex we get

$$\epsilon = i\alpha \qquad \boldsymbol{k}\cdot\boldsymbol{d} = i\beta$$

Using these in (4) and (5) we get

$$\frac{T_t}{T_i} = \left[\frac{2i\alpha}{2i\alpha\cos h\beta + (1-\alpha^2)\sinh\beta}\right]^2 = \frac{4\alpha^2}{4\alpha^2 + (1+\alpha^2)\sinh^2\beta}$$

and smimilarly the ratio of reflected to transmitted power is

$$\frac{T_r}{T_i} = \frac{(1+\alpha^2)^2\sinh^2\beta}{4\alpha^2 + (1+\alpha^2)\sinh^2\beta}$$

Substuting $\beta = kd$ and since $n = 1$ we get $\epsilon = \frac{\cos i}{\cos r}$ we get

$$\frac{T_r}{T_i} = \frac{(1+\alpha^2)^2\sinh^2 kd}{4\alpha^2 + (1+\alpha^2)\sinh^2 kd}$$

Graphing this function as a function of $\frac{d}{\lambda}$ we get.

□