PHYS 512: Electromagnetic Theory II

Homework #1

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1. **(Jackson 6.1)** In three dimensions the solution to the wave equation (6.32) for a point source in space and time (a light flash at t' = 0, x' = 0) is a spherical shell disturbance of radius $R = ct$, namely the Green function $G^{(+)}$. It may be initially surprising that in one or two dimensions, the disturbance possesses a "wake", even though the source is a "point" in space and time. The solutions for fewer dimensions than three can be found by superposition in the superfluous dimension(s), to eliminate dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

   (a) Starting with the retarded solution to the three-dimensional wave equation, show that the source $f(\boldsymbol{x}', t) = \delta(x')\delta(y')\delta(t')$, equivalent to a $t = 0$ point source at the origin in two spatial dimensions, produces a two-dimensional wave,

   $$\Psi(x, y, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}$$

   where $\rho^2 = x^2 + y^2$ and $\Theta(\xi)$ is the unit step function $[\Theta(\xi) = 0(1)$ if $\xi < (>)0]$
   Solution:
   The retarded solution is

   $$\Psi(x, y, z, t) = \int \frac{[f(\boldsymbol{x}', t')]_{\text{ret}}}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x' \tag{1}$$

   Substuting the source function with the given delta functions we get

   $$\Psi = \int \frac{\delta(x')\delta(y')\delta(t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c})}{R} dx'dy'dz'$$

   $$= \int_{-\infty}^{\infty} \frac{\frac{\delta(t-R}{c})}{R} dz'$$

   Since we have cylindrical coordinate system we get

   $$R = |\boldsymbol{x} - \boldsymbol{x}'| = \sqrt{\rho^2 + (z - z')^2} \text{ where } x' = y' = 0$$

   This integral can be done with substitution. Supposing $u = z' + z$, we get $dz' = du$ and the limit stay the same

   $$\Psi(\rho, t) = \int_{-\infty}^{\infty} \frac{\delta(t - \sqrt{\rho^2 + u^2}/c)}{\sqrt{\rho^2 + u^2}} du \tag{2}$$

   Now this integral is of the form

   $$\Psi(a) = \int \frac{\delta(f(x, a))}{g(x)} dx$$

making substitution of variable $f(x) = \beta$ we get $d\beta = f'(x)dx$ so that we get

$$\Psi(a) = \int \frac{\delta(\beta)}{g(x)} \frac{1}{f'(x)} d\beta$$

It is clear that the delta function only picks up values of x for which $\beta = f(x) = 0$. So the delta function reduces the integral to the sum of finite values for which $\beta = f(x) = 0$, let the solutions of $\beta = f(x) = 0$ be α_i, this makes,

$$\Psi(a) = \sum_i \frac{1}{g(\alpha_i) f'(\alpha_i)}$$

for this problem we have $f(u) = t - \frac{\sqrt{\rho^2+u^2}}{c}$ whose zeros are

$$t - \frac{\sqrt{\rho^2 + \alpha_i^2}}{c} = 0 \qquad \Longrightarrow \alpha_i = \pm\sqrt{c^2t^2 - \rho^2} \qquad \text{if } ct > \rho$$

there are no roots if $ct < \rho$ and the delta function is zero and the integral is identically zero. Also the derivative at the root is

$$f'(u) = \frac{u}{c\sqrt{\rho^2 + u^2}} \qquad \Longrightarrow f'(\alpha_i) = \pm\frac{\sqrt{c^2t^2 - \rho^2}}{cct}$$

Substituting this in the integral (2), knowing that there are two values of α_i we get

$$\Psi(\rho, t) = \begin{cases} \frac{2c^2 t}{\sqrt{c^2t^2-\rho^2}} \frac{1}{ct} & \text{if } ct \geq \rho \\ 0 & \text{if } ct \leq \rho \end{cases}$$

the two cases can be combined by using heaviside function

$$\Psi(x, y, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}} = \frac{2c\Theta(ct - \sqrt{x^2 - y^2})}{\sqrt{c^2t^2 - x^2 - y^2}}$$

This is the required form of the wave

□

(b) Show that a "sheet" source, equivalent to a point pulse source at the origin in one space dimension produces a one dimensional wave proportional to

$$\Psi(x, t) = 2\pi c\Theta(ct - |x|)$$

Solution:

For the sheet source we expect a plane propagation of the wave. The source function for the sheet source at some particular time $t' = 0$, let the $x' = 0$ plane be the source, so we can write the source function as

$$f(t', x') = \delta(x')\delta(t')$$

Using this source function to get the retarded time solution and substuting in (1) we get

$$\Psi(x, y, z, t) = \int_{-\infty}^{\infty} \frac{\delta(x')\delta(t')_{\text{ret}}}{R} dx'dy'dz'$$

Again we get $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$. Again similar to the previous rpoblem chaning of variables with $u = y - y', v = z - z'$ and recognizing that the delta function integral simply picks up $x' = 0$ we get

$$\Psi(x, y, z, t) = \int_{-\infty}^{\infty} \frac{\delta(t - \frac{\sqrt{x^2+u^2+v^2}}{c})}{\sqrt{x^2 + u^2 + v^2}} dudv$$

Since the integral has cylindrical symmetry when we have $\rho = \sqrt{u^2+v^2}$ we can make cylindrical variable substitution to get

$$\Psi(\rho,\phi,z) = \int \frac{\delta t - \sqrt{\rho^2+x^2}/c}{\sqrt{\rho^2+x^2}} \rho d\rho d\phi$$

Due to cylindrical independence the phi integral is 2π and we are left with delta function integral similar to previous problem

$$\Psi(\boldsymbol{x},t) = \int \frac{\delta(t - \sqrt{\rho^2+x^2}/c)}{\sqrt{\rho^2+x^2}} \rho d\rho$$

This again has a delta function inside the integral, and is non-zero only for the delta function equal to zero, the zeros of the expression inside the delta function, only give non zero values and the integral truns to a sum over these finite values of solution, the zeros of the delta are

$$t - \sqrt{\rho^2+x^2}/c = 0 \qquad \Longrightarrow \qquad \rho = \pm\sqrt{c^2t^2-x^2} \text{ if } ct > x$$

Also supposing $\beta = f(\rho) = t - \sqrt{\rho^2+x^2}/c$ we get

$$d\beta = f'(\rho)d\rho \qquad d\beta = \frac{2\rho}{2c\sqrt{\rho^2+x^2}} \qquad \Longrightarrow \rho d\rho = c\sqrt{\rho^2+x^2}d\beta$$

Substuting these

$$\Psi(\boldsymbol{x},t) = \int \frac{\delta(\beta)}{\sqrt{\rho^2+x^2}} c\sqrt{\rho^2+x^2}d\beta$$

Since there are two values of zeros of the funtion we have two terms in sum and we get

$$\Psi(\boldsymbol{x},t) = c + c$$

By similar arguments as in the previous one we get non zero integral only if $ct > x$ we can write this using the Heaviside function

$$\Psi(\boldsymbol{x},t) = 2c\Theta(ct - x)$$

This is the required function. □

2. **(Jackson 6.4)** A uniformly magnetized and conducting sphere of radius R and total magnetic moment $m = 4\pi MR^3/3$ rotates about its magnetization axis with angular speed ω . IN the steady state no current flows in the conductor. The motion is non relativistic; the sphere has not excess charge on it.

   (a) By considering Ohm's law in the moving conductor, show that the motion induces and electric field and a uniform volume charge density in the conductor $\rho = m\omega/\pi c^2 R^3$
   Solution:
   The magnetic moment of sphere is given by $\boldsymbol{m} = \boldsymbol{M}V$ where $V = \frac{2}{3}\pi R^3$ is the volume of sphere. Comparing it to the given magnetic moment we get that $\boldsymbol{M} = M\hat{\boldsymbol{z}}$. The magnetic flux density inside the sphere is given by

$$\boldsymbol{B} = \frac{2}{3}\mu_0 \boldsymbol{M} = \frac{\mu_0 m}{2\pi R^3}\hat{\boldsymbol{z}}$$

By ohm's law the current in the moving conductor is

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

Since there is no current $J = 0$ which implies

$$\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$$

Since the sphere has angular frequency $\boldsymbol{\omega}$, the translational velocity at r is given by $\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{\omega} = \omega \boldsymbol{r} \times \hat{\boldsymbol{z}}$ thus we get

$$\boldsymbol{E} = \boldsymbol{r} \times \boldsymbol{\omega} \times \boldsymbol{B} = \frac{\mu_0 m}{2\pi R^3} \left[\hat{\boldsymbol{z}}(\hat{\boldsymbol{z}} \cdot \boldsymbol{r}) - \boldsymbol{r}(\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}})\right]$$

This simplifies to

$$\boldsymbol{E} = \frac{\mu_0 m\omega}{2\pi R^3} (\hat{\boldsymbol{z}}(\hat{\boldsymbol{z}} \cdot \boldsymbol{r}) - \boldsymbol{r})$$

This is the projection of vector $\boldsymbol{r}$ onto the horizontal axis, which in cylindrical system is

$$E_\rho = -\frac{\mu_0 m\omega\rho}{2\pi R^3}$$

Now that we have the field we can apply gauss' law to calculate the charge density

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

Sinc our field only has component along ρ we have

$$\rho = \epsilon_0 \frac{\partial E_\rho}{\partial \rho} = -\frac{\mu_0 \omega m \rho}{2\pi R^3}$$

This is the required volume charge density. □

(b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry arguments to shwo that the lowest possible electric multipolarity is quadrupole. Show that only quadrupole field exists outside that the quadrupule moment tensor has non vanishing components $Q_{33} = -4m\omega R^2/3c^2$, $Q_{11} = Q_{22} = \frac{-Q_{33}}{2}$.

Solution:

Since there is no charge inside the sphere the exterior can be described as the multipole expansion. Since there is no charge, the monopole moment which is the moment of total charge is zero. The electrostatic potential can be obtained as

$$\Phi(\rho) = -\int \boldsymbol{E} d\boldsymbol{l} = -\int E_\rho d\rho = \Phi_0 + \frac{\mu_0 m\omega\rho^2}{4\pi R^3}$$

This can be simplified by using the cartesian coordinate formulation as

$$\Phi(r,\theta) = \Phi_0 + \frac{\mu_0 m\omega}{4\pi R^3} r^2 \sin^2\theta.$$

Wrinting $\sin^2\theta$ in terms of legendre polynomials we get

$$\Phi(r,\theta) = \Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2 \left[P_0(\cos\theta) - p_2(\cos\theta)\right]$$

this simlifies to

$$\Phi(r,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2\right) P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R^3} r^2 P_2(\cos\theta)$$

At the surface of the sphere $r = R$ we get the potential as

$$\Phi(r,\theta) = \left(\Phi_0 + \frac{\mu_0 m\omega}{6\pi R^3} r^2\right) P_0(\cos\theta) - \frac{\mu_0 m\omega}{6\pi R^3} r^2 P_2(\cos\theta)$$

Since the potential is azimuthally symmetric, we can write the external potential as a legendre polynomial series

$$V(\theta) = \sum_i A_l P_l(\cos\theta)$$

on the surface, and out side the surface the potential is

$$\Phi(,\theta) = \sum_l A_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)$$

Since there is no charge the monopole term for $l = 0$ vanishes so we get

$$\Phi_0 = -\frac{\mu_0 m\omega}{6\pi R}$$

And the expression becomes.

$$\Phi(r,\theta) = -\frac{\mu_0 m\omega R^2}{6\pi r^3} P_2(\cos\theta)$$

Now tath we hae te exterior potential can be converted to expression with spherical harmonics

$$\Phi = -\sqrt{\frac{4\pi}{5}} \frac{\mu_0 m\omega R^2}{6\pi} \frac{Y_{20}(\theta,\phi)}{r^2}$$

The standard multipole expansion expression is

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{l=-\infty}^{\infty} \sum_{m=-l}^{l} \frac{2\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

compariosion gives

$$q_{20} = -4\pi\epsilon_0 \sqrt{\frac{5}{4\pi}} \frac{\mu_0 m\omega R^2}{6\pi} = -\sqrt{\frac{5}{4\pi}} \frac{2m\omega R^3}{3c^2}$$

The moment expression in cartesian coordinate system is given by

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}} q_{20} = -\frac{4m\omega R^2}{3c^2}, Q_{11} = Q_{22} = -\frac{1}{2} Q_{33}$$

this is teh required expression. □

(c) By considering the radial electric fields inside and outside the sphere, show that the necessary surface charge density $\sigma(\theta)$ is

$$\sigma(\theta) = \frac{1}{4\pi R^2} \cdot \frac{4m\omega}{3c^2} \cdot \left[1 - \frac{5}{2} P_2(\cos\theta)\right]$$

Solution:
the surface charge can be computed by using teh normal component as derivatives of potential. In the spherical coordinates we get

$$E_r^{\text{out}} = -\frac{\mu_0 m\omega R^2}{2\pi r^4} P_2(\cos\theta)$$

$$E_r^{\text{in}} = -\frac{\mu_0 m\omega r}{3\pi R^3} \left[p_o(\cos\theta) - P_2(\cos\theta)\right]$$

the surface charge is thus

$$\sigma = \epsilon_0 \left(E_r^{\text{out}} - E_r^{\text{in}}\right)\Big|_{r=R} = \frac{\mu_0\epsilon_0 m\omega}{3\pi R^2} \left[\frac{3}{2} P_2(\cos\theta) - (P_0(\cos\theta) - P_2(\cos\theta))\right]$$

$$= \frac{m\omega}{3\pi c^2 R^3} \left[P_0(\cos\theta) - \frac{5}{2} P_2(\cos\theta)\right]$$

This gives the required expression for teh surface charge density. □

(d) The rotating sphere serves aas a unipolar induction devie if a stationary circuit is attached by a slip ring to the pole and sliding contact to the equator. Show that hte line integral of the electric field from the equator contat to the pole contact is $\mathcal{E} = \mu_0 m\omega/4\pi R$

Solution:

The line integral is

$$\mathcal{E} = \int_{\text{equator}}^{\text{pol}} \boldsymbol{E}d\boldsymbol{l} = \Phi_{\text{equator}} - \Phi_{\text{pol}} = \Phi(\theta = \pi/2) - \Phi(\theta = 0)$$

Substuting the value of theta in the expression for the potential we get

$$\mathcal{E} = -\frac{\mu_0 m\omega}{6\pi R}\left[P_2(0) - P_2(1)\right] = \frac{\mu_0 m\omega}{4\pi R}$$

This gives the required expression for the integral.

□