PHYS 511: Electrodynamics

Homework #5

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1. **(Jackson 3.6)** Two point charges q and $-q$ are located on the z axis at $z = a$ and $z = -a$ respectively

   (a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for both $r > a$ and $r < a$.

   Solution:
   Le the position vector of point charges $+q$ and $-q$ be $\boldsymbol{r}_1(a, 0, \phi)$ and $\boldsymbol{r}_2(-a, \pi, \phi)$ respectively. Any point with position vector $\boldsymbol{r}$ will have potential given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|} - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_2|}\right]$$

   If the angle between two position vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ is γ, a function of this form, with the help of cosine law, can be written as

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \begin{Bmatrix} \frac{1}{r'\sqrt{1+\left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\gamma}} & \text{if } r' \geq r \\ \frac{1}{r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\gamma}} & \text{if } r' < r \end{Bmatrix} = \sum_{n=0}^{\infty}\left(\frac{r_<^n}{r_>^{n+1}}\right)P_n(\cos\gamma)$$

   Here, $r_> = \max(r, r')$ and $r_< = \min(r, r')$. Also the generating function expansion of legendre polynomials has been used

$$\forall\, t < 1 : \frac{1}{\sqrt{1 + t^2 - 2tx}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

   By using the addition theorem for the legendre polynomials we can write

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta_1, \phi_1)^* Y_l^m(\theta, \phi)$$

   So we can write the expression

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|} = \sum_{l=0}^{\infty}\frac{r^n}{r_1^{n+1}}\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta_1, \phi_1)^* Y_l^m(\theta, \phi)$$

   Since we have $|\boldsymbol{r}_1| = |\boldsymbol{r}_2| (= a)$, we can generalize $r_> = \max(r, a)$ and $r_< = \min(r, a)$. So the potnetial expression becomes

$$\begin{aligned} \Phi &= \frac{q}{4\pi\epsilon_0}\sum_{l=o}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left\{Y_l^m(\theta_1, \phi_1)^* Y_l^m(\theta, \phi) - Y_l^m(\theta_2, \phi_2)^* Y_l^m(\theta, \phi)\right\} \\ &= \frac{q}{4\pi\epsilon_0}\sum_{l=o}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left[\; Y_l^m(0, \phi)^* - Y_l^m(\pi, \phi)^*\right] Y_l^m(\theta, \phi) \end{aligned}$$

For the given problem $\theta_1 = 0, \theta_2 = \pi$. But

$$\forall m \neq 0 : Y_l^m(0,\phi) = 0 \quad \wedge \quad Y_l^0(0,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(1) \quad \Rightarrow Y_l^m(0,\phi) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m,0}$$
$$\forall m \neq 0 : Y_l^m(\pi,\phi) = 0 \quad \wedge \quad Y_l^0(\pi,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(-1) \quad \Rightarrow Y_l^m(\pi,\phi) = \sqrt{\frac{2l+1}{4\pi}} (-1)^l \delta_{m,0}$$

Substuting these we get

$$\begin{aligned}
\Phi &= \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} \frac{r_<^l}{r_>^{l+1}} \left[\ (1-(-1)^l)\delta_{m,0} \right] Y_l^m(\theta,\phi) \\
&= \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sqrt{\frac{4\pi}{2l+1}} \frac{r_<^l}{r_>^{l+1}} (1-(-1)^l) Y_l^0(\theta,\phi)
\end{aligned}$$

Since $Y_l^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$ and $\forall k \in \mathbb{N} : (1-(-1)^{2k} = 0) \wedge (1-(-1)^{2k+1} = 2)$, we get

$$\forall k \in \mathbb{N} : \Phi = \frac{2q}{4\pi\epsilon_0} \sum_{k=0}^{\infty} \left(\frac{r_<^{2k+1}}{r_>^{2k+2}} \right) P_{2k+1}(\cos\theta) = \begin{cases} \frac{q}{2\pi\epsilon_0} \sum_{k=0}^{\infty} \left(\frac{r^{2k+1}}{a^{2k+2}} \right) P_{2k+1}(\cos\theta) & \text{if } r \leq a \\ \frac{q}{2\pi\epsilon_0} \sum_{k=0}^{\infty} \left(\frac{a^{2k+1}}{r^{2k+2}} \right) P_{2k+1}(\cos\theta) & \text{if } r > a \end{cases}$$

This is the required expression for teh potential due to this dipole. □

(b) Keeping the product $qa = p/2$ constant, take the limit of $a \to 0$ and find the potential for $r \neq 0$. This is by definition a dipole along the z axis and its potential.
Solution:
In the limit $a \to 0$ we have $r > a$ so we get

$$\begin{aligned}
\Phi &= \lim_{a\to 0} \frac{q}{2\pi\epsilon_0} \sum_{k=0}^{\infty} \left(\frac{a^{2k+1}}{r^{2k+2}} \right) P_{2k+1}(\cos\theta) \\
&= \lim_{a\to 0} \frac{qa}{2\pi\epsilon_0} \left(\frac{1}{r^2} P_1(\cos\theta) + \frac{a^2}{r^3} P_3(\cos\theta) + \dots \right) \\
&= \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}
\end{aligned}$$

This is the required expression for potential due to a dipole. □

(c) suppose now that the dipole in (1b) is surrounded by a grounded spherical shell of radius b concentric with the origin. By linear superposition find the potential everywhere inside the shell.
Solution:
Since the grounded sphere attains charge due to induction of the dipole inside it. It creates its own electric potential inside the sphere which follows Laplace's equation.The general solution to Laplace's equation in spherical coordinate system is

$$u(r,\theta,\phi) = [Ar^l + Br^{-(l+1)}][C\cos m\phi + D\sin m\phi][EP_l^m(\cos\theta) + FQ_l^m(\cos\theta)]$$

Since there is azimuthal symmetry the value of $m = 0$. The potential is finite at both the poles, but the associated Legendre function of second kind $Q_l^m(x)$ diverges at $x = \pm 1$, which corresponds to poles, so we require $F = 0$. Also since the potential is finite at $r = 0$ we require $B = 0$ Absorbing constant E into A_k, the general solution reduces to

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{1}$$

Here the function $P_l^0(x) = P_l(x)$ is the Legendre polynomial. By superposition principle the totential inside the sphere of radius b must be potential due to the induced charge in sphere and the potential by dipole. So potential everywhere inside the sphere is

$$\Phi' = \Phi + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = \frac{p}{4\pi\epsilon_0}\frac{P_1(\cos\theta)}{r^2} + \sum_{l=0}^{\infty} A r^l P_l(\cos\theta)$$

But we require $\Phi' = 0$ at $r = b$.

$$\sum_{l=0}^{\infty} A_l b^l P_l(\cos\theta) = -\frac{q}{4\pi\epsilon_0}\frac{P_1(\cos\theta)}{b^2}$$

Since $\{P_l(x); l \in \mathbb{N}\}$ form a set of orthogonal functions the coefficient of $P_l(x)$ on either side of equation must be equal for this equation to be identity, thus we get

$$A_1 b = -\frac{q}{4\pi\epsilon_0}\frac{1}{b^2} \implies A_1 = -\frac{q}{4\pi\epsilon_0}\frac{1}{b^3}; \qquad A_l b^l = 0, \implies A_l = 0; \forall\, l \neq 1$$

Using the value of A_l in (1) we get

$$\Phi' = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} - \frac{q}{4\pi\epsilon_0 b^3} r\cos\theta = \frac{1}{4\pi\epsilon_0}\left[\frac{p}{r^2} - \frac{r}{b^3}\right]\cos\theta$$

This is the required potential everywhere inside the sphere □

2. **(Jackson 4.1)** Try to obtain results for the non vanishing moments valid or all l, but in each case find the first two sets of non vanishing moments at the very least. Calculate the multipole moments q_{lm} of the charge distributions shown

 (a)

 Solution:
 The charge density can be written as

$$\rho(\boldsymbol{x}) = \frac{q}{r^2}\delta(r-a)\delta(\cos\theta)\left[\delta(\phi) + \delta\left(\phi + \frac{\pi}{2}\right) - \delta(\phi - \pi)\delta\left(\phi + \frac{3\pi}{2}\right)\right]$$

 Since all the charges are in plane $\theta = \frac{\pi}{2}$ so $\cos\theta = 0$. The multipole moments are given by

$$\begin{aligned} q_{lm} &= \int r^l Y_l^m(\theta,\phi)\rho(\boldsymbol{x}) d^3x \\ &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} q a^l P_l^m(0)\left[1 + e^{-im\pi/2} - e^{-im\pi} - e^{-im3\pi/2}\right] \end{aligned}$$

 Sicne $P_l^m(0) = 0$ for all even m we can write $m = 2k+1; k \in \mathbb{N}$

$$\begin{aligned} q_l^{2k+1} &= 2qa^l\left[1 - i(-1)^k\right]\sqrt{\frac{2l+1}{4\pi}\frac{(l-(2k+1))!}{(l+(2k+1))!}} P_l^{2k+1}(0) \\ &= 2qa^l\left[1 - i(-1)^k\right] Y_l^{2k+1}\left(\frac{\pi}{2}, 0\right) \end{aligned}$$

 This vanishes for all even l thus the values for odd l and m are

$$\begin{aligned} q_{1,1} &= -q_{1,-1}^* = -2qa(1-i)\sqrt{\frac{3}{8\pi}} \\ q_{3,3} &= -q_{3,-3}^* = 2qa^3(1+i)\sqrt{\frac{35}{4\pi}} \\ q_{3,1} &= -q_{3,-1}^* = 2qa^3(1-i)\frac{1}{4}\sqrt{\frac{21}{4\pi}} \end{aligned}$$

 Theseare the first few non vanishing moments. □

(b)

Solution:
The charge ensity is

$$\rho(\boldsymbol{x}) = \frac{q}{2\pi r^2}\left[\delta(r-a)\delta(1-\cos\theta) + \delta(r-a)\delta(1+\cos\theta) - \delta(r)\right]$$

The multipole moments are given by

$$\begin{aligned} q_{lm} &= \int r^l Y_l^m(\theta,\phi)\rho(\boldsymbol{x})d^3x \\ &= qa^l P_l^m(0)\left[Y_l^m(0,0)^* + Y_l^m(\pi,0)^*\right] \end{aligned}$$

for $l > 0$ and $q_{00} = 0$. By azimuthal symmetry, only the $m = 0$ moments are non vanishing. Thus we get

$$\begin{aligned} q_{l_0} &= qa^l\sqrt{\frac{2l+1}{4\pi}}\left[P_l(1) + P_(-1)\right] \\ &= qa^l\left[1+(-1)^l\right]\sqrt{\frac{2l+1}{4\pi}} \qquad l > 0 \end{aligned}$$

So, this leads to

$$\begin{aligned} q_{2,0} &= \sqrt{\frac{5}{\pi}}qa^2; \qquad q_{2,m\neq 0} = 0 \\ q_{4,0} &= \sqrt{\frac{9}{\pi}}qa^4; \qquad q_{4,m\neq 0} = 0 \end{aligned}$$

These are the moments. □

(c) For the charge distribution of the second set b write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the x_0y plane as a function of distance from the origin for the distances grater than a.

Solution:
The expansion of the potential in terms of multipole coefficients is

$$\Phi = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

Since we only have non-zero coefficients for $m = 0$ and l even we have

$$\begin{aligned} \Phi &= \frac{1}{4\pi\epsilon_0}\sum_{l=2,2,4}\frac{4\pi}{2l+1}q_{l0}\frac{Y_l^0(\theta,\phi)}{r^{l+1}} \\ &= \frac{q}{4\pi\epsilon_0}\sum_{l=2,4\ldots}\frac{4\pi}{2l+1}qa^l\sqrt{\frac{2l+1}{\pi}}\sqrt{\frac{2l+1}{\pi}}\frac{P_l(\cos\theta)}{r^{l+1}} \\ &= \frac{q}{4\pi\epsilon_0}2\frac{a^l}{r^l+1}P_l(\cos\theta) \end{aligned}$$

The lowest order term is $l = 2$. And in the $x - y$ plane $\theta = \frac{\pi}{2}$ so we get

$$\Phi = -\frac{q}{4\pi\epsilon_0 a}\left(\frac{a}{r}\right)^3$$

This is the inverse cubic function whose graph is shown in Fig. (1) looks like. □

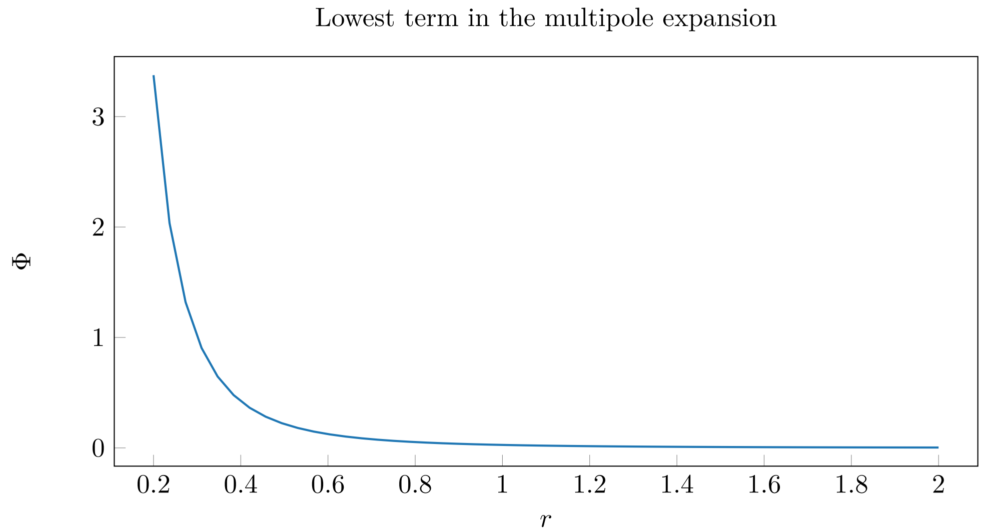

Figure 1: First term of multipole expansion.

(d) Calculate directly from Coloumb's law the exact potential for b in the $x-y$ plane. Plot it as a function of distance and compare withe the result found in part c.
Solution:
For the charges given we have in the cartesian coordinate system, in $x-y$ plane, if the distance from the origin to any point on the plane is r we get

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+a^2}} - \frac{1}{r} + \frac{1}{\sqrt{r^2+a^2}}\right)$$

Plotting this as a function r we get the plot in Fig. (2) □

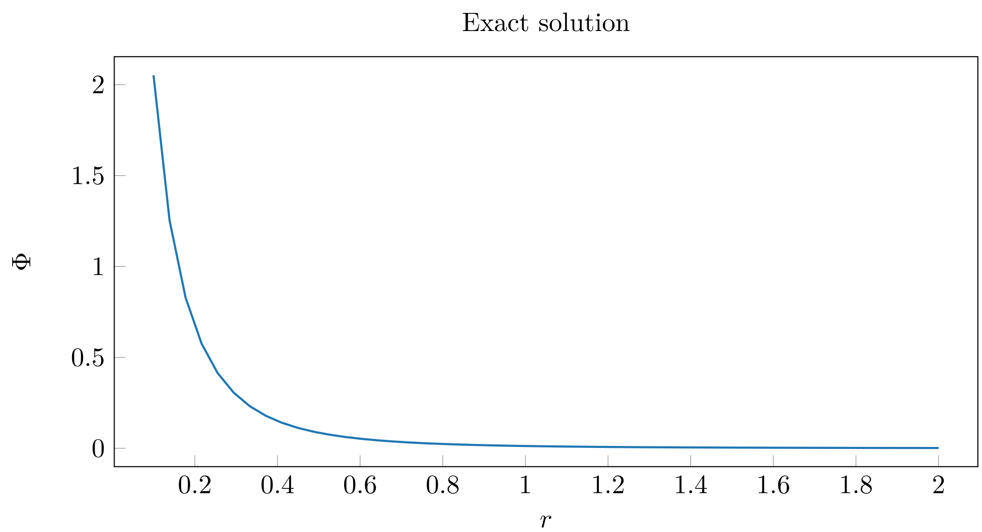

Figure 2: Exact solution

3. **(Jackson 4.9)** A point charge q is located in free space a distance d from the center of a dielectric sphere of radius a ($a < d$) and dielectric constant ϵ/ϵ_0

(a) Find the potential at all points in space as an expansion in spherical harmonics.
Solution:

The charge at d inuces charge in the sphere. The induced charge produces the field inside the sphere. Again, using the general solution of Lapalace's equation in spherical system with azimuthal symetry we get

$$\Phi_{in} = \frac{q}{4\pi\epsilon}\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{2}$$

We can chose the coordinate system such that the Z axis of our coordinate system passes through the charge and the center of sphere. WIh this. Outside the sphere the potential due to the chage is given by

$$\begin{aligned}\Phi_{out} &= \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty} B_l \frac{a^l}{r^{l+1}} P_l(\cos\theta) &(3)\\ &= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} + B_l\left(\frac{a^l}{r^{l+1}}\right)\right] P_l(\cos\theta) &(4)\end{aligned}$$

The component of electri field parallel to the surface of the sphere is

$$E_\theta^{\text{in}} = -\left.\frac{1}{r}\frac{\partial\Phi_{\text{in}}}{\partial\theta}\right|_{r=a} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[A_l \frac{r^l}{a^{l+1}} P_l'(\cos\theta)\sin\theta\right]_{r=a} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty} A_l \frac{1}{a} P_l'(\cos\theta)\sin\theta \tag{5}$$

Similarly the component outside the sphere is

$$E_\theta^{\text{out}} = \left.\frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} + B_l\left(\frac{a^l}{r^{l+1}}\right)\right] P_l'(\cos\theta)\sin\theta\right|_{r=a} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{a^l}{d^{l+1}} + \frac{B_l}{a}\right] P_l'(\cos\theta)\sin\theta \tag{6}$$

Equating (5) and (6) we get

$$\frac{q}{4\pi\epsilon}\frac{A_l}{a} = \frac{q}{4\pi\epsilon_0}\left[\frac{a^l}{d^{l+1}} + \frac{B_l}{a}\right] \implies A_l = \frac{\epsilon}{\epsilon_0}\left[\frac{a^{l+1}}{d^{l+1}} + B_l\right] \tag{7}$$

$$E_r^{\text{in}} = -\epsilon\left.\frac{\partial\Phi_{\text{in}}}{\partial r}\right|_{r=a} = \left.\frac{q}{4\pi}\sum_{l=0}^{\infty}\left[A_l \frac{l r^{l-1}}{a^{l+1}}\right] P_l(\cos\theta)\right|_{r=a} = \frac{q}{4\pi}\sum_{l=0}^{\infty}\left[A_l \frac{l}{a^2}\right] P_l(\cos\theta) \tag{8}$$

Similarly for the radial component of field outsie the sphere is

$$E_r^{\text{out}} = -\epsilon_0\left.\frac{\partial\Phi_{\text{out}}}{\partial r}\right|_{r=a} = \left.\frac{q}{4\pi}\sum_{l=0}^{\infty}\left[\frac{l a^{l-1}}{d^{l+1}} - B_l\frac{(l+1)a^{l+1}}{r^{l+2}}\right] P_l(\cos\theta)\right|_{r=a} = \frac{q}{4\pi}\left[A_l\frac{l a^{l-1}}{d^{l+1}} - B_l\frac{(l+1)}{a^2}\right] P_l(\cos\theta) \tag{9}$$

Equating (8) and (9) we get

$$\frac{q}{4\pi}\frac{A_l l}{a^2} = \frac{q}{4\pi}\left[\frac{l a^{l-1}}{d^{l+1}} - B_l\frac{l+1}{a^2}\right] = A_l \implies \frac{a^{l+1}}{d^{l+1}} - B_l\frac{l+1}{l} \tag{10}$$

Solving two linear equations in A_l and B_l from (10) and (7) we get

$$\begin{aligned}B_l &= \frac{\left(\frac{\epsilon_0}{\epsilon} - 1\right) l}{l + (l+1)\frac{\epsilon_0}{\epsilon}}\frac{a^{l+1}}{d^{l+1}} &(11)\\ A_l &= \frac{2l+1}{l + (l+1)\frac{\epsilon_0}{\epsilon}}\frac{a^{l+1}}{d^{l+1}} &(12)\end{aligned}$$

Substuting the coefficient in (11) an (3) we get

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon}\sum_{l=0}^{\infty}\frac{2l+1}{l + (l+1)\frac{\epsilon_0}{\epsilon}}\frac{r^l}{d^{l+1}} P_l(\cos\theta)$$

And similarly

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r^l}{d^{l+1}} + \frac{(\frac{\epsilon_0}{\epsilon}-1)l}{l+(1+l)\frac{\epsilon}{\epsilon_0}}\frac{a^{2l+1}}{(rd)^{l+1}}\right]P_l(\cos\theta)$$

These are the expression for the electri field inside and outside the sphere. □

(b) Calculate the rectangular components of the electric field near the center of the sphere.
Solution:
Inside the sphere, the first few terms are

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\frac{\epsilon_0}{\epsilon}}P_0(\cos\theta) + \frac{3}{1+2\frac{\epsilon_0}{\epsilon}}\frac{r}{d}P_1(\cos\theta) + \frac{5}{2+3\frac{\epsilon_0}{\epsilon}}\frac{r^2}{d^2}P_2(\cos\theta) + \mathcal{O}(r^3)\right]$$

The radial radial component of the field is

$$\boldsymbol{E}_r = -\frac{\partial\Phi_{\text{in}}}{\partial r}\hat{\boldsymbol{r}} = -\frac{q}{4\pi\epsilon}\left[0 + \frac{3}{1+2\frac{\epsilon_0}{\epsilon}}\frac{1}{d}P_1(\cos\theta) + \mathcal{O}(r)\right]\hat{\boldsymbol{r}} = -\frac{q}{4\pi\epsilon d}\left[\frac{3\cos\theta}{1+2\frac{\epsilon_0}{\epsilon}} + \mathcal{O}(r)\right]\hat{\boldsymbol{r}}$$

In the limit $r \to 0$ we get

$$\boldsymbol{E}_r = -\frac{q}{4\pi\epsilon d}\left[\frac{3\cos\theta}{1+e\frac{\epsilon_0}{\epsilon}}\right]\hat{\boldsymbol{r}}$$

Similarly the tangential (θ) component of field is

$$\boldsymbol{E}_\theta = -\frac{1}{r}\frac{\partial\Phi_{\text{in}}}{\partial\theta}\hat{\boldsymbol{\theta}} = \frac{-1}{r}\frac{q}{4\pi\epsilon}\left[0 + \frac{-3\sin\theta}{1+e\frac{\epsilon_0}{\epsilon}}\frac{r}{d} + \mathcal{O}(r)\right]\hat{\boldsymbol{\theta}} = \frac{q}{4\pi\epsilon d}\left[\frac{3\sin\theta}{1+2\frac{\epsilon_0}{\epsilon}} + \mathcal{O}(r)\right]\hat{\boldsymbol{\theta}}$$

In the limit $r \to 0$ we get

$$\boldsymbol{E}_\theta = \frac{q}{4\pi\epsilon d}\left[\frac{3\sin\theta}{1+2\frac{\epsilon_0}{\epsilon}}\right]\hat{\boldsymbol{\theta}}$$

Since the ϕ component of the field is 0 as the potential is independent of ϕ we get

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon d}\frac{3}{1+2\frac{\epsilon_0}{\epsilon}}\left[-\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right] = \frac{q}{4\pi\epsilon d}\frac{3}{1+2\frac{\epsilon_0}{\epsilon}}\hat{\boldsymbol{k}}$$

Where $\hat{\boldsymbol{k}}$ is the unit vector along $z-$axis. □

(c) Verify that, in this limit $\epsilon/\epsilon_0 \to \infty$, hour result is the same as that for conducting sphere
Solution:
In the limit $\epsilon/\epsilon_0 \to \infty$ we have

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon_0 d}$$

and

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0}\left[\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} - \sum_{l=1}^{\infty}\frac{a^{2l+1}}{(rd)^{l+1}}\right]P_l(\cos\theta)$$

We can invoke the spherical harmonics expansion in reerse and write the expression as

$$\frac{q}{4\pi\epsilon_0}\left[\frac{q/d}{r} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} - \frac{a}{|d\boldsymbol{r} - a^2\hat{\boldsymbol{r}}|}\right]$$

Whih is indeed the potential of a sphere outside the sphere □