PHYS511: Electrodynamics

Homework #4

Prakash Gautam

February 7, 2019

1. **(Jackson 3.1)** Two concentric spheres have radii a , b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and teh lower hemisphere of the outer sphere aare maintained at potential V . The other hemispheres are at zero potential.

   Determinte the potential int eh region $a \leq r \leq b$ as the series in Legendre polynomials. Include terms at least upt to $l = 4$. Check your solution against known results in the limiting chases $b \to \infty$, and $a \to 0$.
   Solution:
   The general solution to Laplace's equation in spherical coordinate system is

$$u(r, \theta, \phi) = [Ar^l + Br^{-(l+1)}][C \cos m\phi + D \sin m\phi][EP_l^m(\cos\theta) + FQ_l^m(\cos\theta)]$$

Since there is azimuthal symmetry the value of $m = 0$. The potential is finite at both the poles, but the associated Legendre function of second kind $Q_l^m(x)$ diverges at $x = \pm 1$, which corresponds to poles, so we require $F = 0$. Absorbing constant C and F into A_k and B_k, the general solution reduces to

$$u(r, \theta, \phi) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta) \tag{1}$$

Here the function $P_l^0(x) = P_l(x)$ is the Legendre polynomial.

Multiplying both sides by $P_k(\cos\theta)$ and integrating with respect to $d\cos\theta$ from -1 to 1 we get

$$\begin{aligned}
\int_{-1}^{1} u(r, \theta, \phi) P_k(\cos\theta) d\cos\theta &= \int_{-1}^{1} \sum_{l=0}^{\infty} [A_k r^l + B_k r^{-(l+1)}] P_l(\cos\theta) P_k(\cos\theta) d\cos\theta \\
&= \sum_{l=0}^{\infty} \left[A_k r^l + B_k r^{-(l+1)}\right] \frac{2}{2l+1} \delta_{lk} \\
&= \left[A_k r^k + B_k r^{-(k+1)}\right] \frac{2}{2k+1}
\end{aligned}$$

Now evaluating the integral for $r = a$ and $r = b$ respectively.

For $r = a$

$$
\begin{aligned}
\frac{2}{2k+1}[A_k a^k + B_k a^{-(k+1)}] &= \int_{-1}^{1} u(a,\theta,\phi) P_k(\cos\theta) d\cos\theta \\
&= \int_{-1}^{0} 0 \cdot P_k(\cos\theta) d\cos\theta + \int_{0}^{1} V P_k(\cos\theta) d\cos\theta \\
&= \int_{0}^{1} V P_k(x) dx \\
&= \frac{V}{2k+1}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}\right]
\end{aligned}
$$

Which implies

$$
A_k a^k + B_k a^{-(k+1)} = \frac{V}{2}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}\right] = \beta \text{ say} \tag{2}
$$

Again doing this for $r = b$ we get

$$
\begin{aligned}
\frac{2}{2k+1}[A_k b^k + B_k b^{-(k+1)}] &= \int_{-1}^{1} u(b,\theta,\phi) P_k(\cos\theta) d\cos\theta \\
&= \int_{-1}^{0} V \cdot P_k(\cos\theta) d\cos\theta + \int_{0}^{1} 0 \cdot P_k(\cos\theta) d\cos\theta \\
&= \int_{0}^{1} V P_k(-x) dx \\
&= \int_{0}^{1} V(-1)^k P_k(x) dx \\
&= \frac{V(-1)^k}{2k+1}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}\right]
\end{aligned}
$$

Which implies

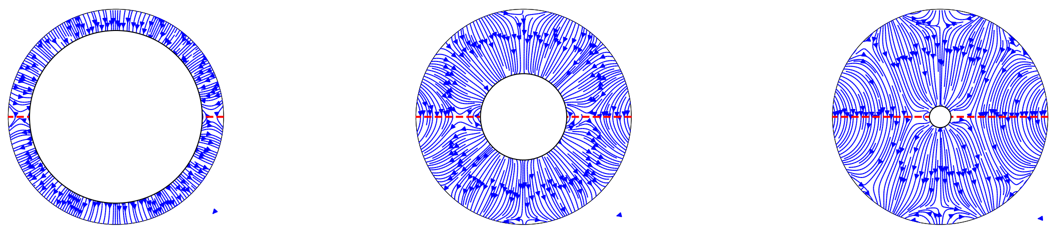
$$
A_k b^k + B_k b^{-(k+1)} = \frac{V(-1)^k}{2}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}\right] = (-1)^k \beta
$$

So the two linear equations are

$$
\begin{aligned}
A_k a^k + B_k a^{-(k+1)} &= \beta \\
A_k b^k + B_k b^{-(k+1)} &= (-1)^k \beta
\end{aligned}
$$

We can cast these two equation of unknowns A_k and B_k into matrix equation as

$$
\begin{bmatrix} a^k & a^{-(k+1)} \\ b^k & b^{-(k+1)} \end{bmatrix} \begin{bmatrix} A_k \\ B_k \end{bmatrix} = \begin{bmatrix} \beta \\ (-1)^k \beta \end{bmatrix} \tag{3}
$$



(a) Eletric field lines with b/a = 1.25 (b) Eletric field lines with b/a = 2.5 (c) Eletric field lines with b/a = 10

Solving the matrix equation we get the matrix

$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = \begin{bmatrix} \frac{\beta\left(a^{k+1}-(-1)^k b^{k+1}\right)}{a^{2k+1}-b^{2k+1}} \\ \frac{\beta(ab)^{k+1}\left(a^k-(-1)^k b^k\right)}{a^{2k+1}-b^{2k+1}} \end{bmatrix}$$

Substuting the value of β from (2) we get

$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = \frac{V}{2} \begin{bmatrix} \frac{\Gamma(\frac{1}{2})\left(a^{k+1}+(-b)^{k+1}\right)(\Gamma(-\frac{k}{2})\Gamma(\frac{k}{2}+\frac{3}{2})-\Gamma(-\frac{k}{2}+1)\Gamma(\frac{k}{2}+\frac{1}{2}))}{(a^{2k+1}-b^{2k+1})\Gamma(-\frac{k}{2})\Gamma(-\frac{k}{2}+1)\Gamma(\frac{k}{2}+\frac{1}{2})\Gamma(\frac{k}{2}+\frac{3}{2})} \\ -\frac{\Gamma(\frac{1}{2})(ab)^{k+1}\left(b^k-(-a)^k\right)(\Gamma(-\frac{k}{2})\Gamma(\frac{k}{2}+\frac{3}{2})-\Gamma(-\frac{k}{2}+1)\Gamma(\frac{k}{2}+\frac{1}{2}))}{(a^{2k+1}-b^{2k+1})\Gamma(-\frac{k}{2})\Gamma(-\frac{k}{2}+1)\Gamma(\frac{k}{2}+\frac{1}{2})\Gamma(\frac{k}{2}+\frac{3}{2})} \end{bmatrix}$$

The coefficients are all zeros for all even $k \geq 2$.

$$\begin{bmatrix} A_{2m} \\ B_{2m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \forall m \in \mathbb{Z}_+;$$

The first few odd of this coefficient are

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = V \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}; \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = V \begin{bmatrix} \frac{3(a^2+b^2)}{4(a^3-b^3)} \\ \frac{-3a^2b^2(a+b)}{4(a^3-b^3)} \end{bmatrix}; \begin{bmatrix} A_3 \\ B_3 \end{bmatrix} = V \begin{bmatrix} \frac{-7(a^4+b^4)}{16(a^7-b^7)} \\ \frac{7a^4b^4(a^3+b^3)}{16(a^7-b^7)} \end{bmatrix}; \begin{bmatrix} A_5 \\ B_5 \end{bmatrix} = V \begin{bmatrix} \frac{11(a^6+b^6)}{32(a^{11}-b^{11})} \\ \frac{-11a^6b^6(a^5+b^5)}{32(a^{11}-b^{11})} \end{bmatrix}$$

Substuting these coefficients in (1) we obtain the potential in with this boundary condition.

$$u(r,\theta,\phi) = V\left[\frac{1}{2} + \frac{3}{4}\left(\frac{r(a^2+b^2)}{a^3-b^3} - \frac{a^2b^2(a+b)}{r^2(a^3-b^3)}\right)P_1(\cos\theta) + \frac{7}{16}\left(-\frac{r^3(a^4+b^4)}{a^7-b^7} + \frac{a^4b^4(a^3+b^3)}{r^4(a^7-b^7)}\right)P_3(\cos\theta)\right.$$
$$\left. + \frac{11}{32}\left(\frac{r^5(a^6+b^6)}{a^{11}-b^{11}} - \frac{a^6b^6(a^5+b^5)}{r^6(a^{11}-b^{11})}\right)P_5(\cos\theta) + \frac{75}{256}\left(-\frac{r^7(a^8+b^8)}{a^{15}-b^{15}} + \frac{a^8b^8(a^7+b^7)}{r^8(a^{15}-b^{15})}\right)P_7(\cos\theta) + \cdots\right]$$

This is the required potential in the region $a \leq r \leq b$. In the limit $b \to \infty$ we have

$$u(r,\theta,\phi) = V\left[\frac{1}{2} + \frac{3}{4}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{16}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{32}\left(\frac{a}{r}\right)^6 P_5(\cos\theta) - \frac{75}{256}\left(\frac{a}{r}\right)^8 P_7(\cos\theta) + \dots\right]$$

In the limit $b \to \infty$ the problem is they potential of splatted sphere everywhere outside the sphere. And the above expression matches the expected result. In the limit $a = 0$, which corresponds to the potential inside the sphere inside the splitted potential.

$$u(r,\theta,\phi) = V\left[\frac{1}{2} - \frac{3}{4}\frac{r}{b}P_1(\cos\theta) + \frac{7}{16}\left(\frac{r}{b}\right)^3 P_3(\cos\theta) - \frac{11}{32}\left(\frac{r}{b}\right)^5 P_5(\cos\theta) + \frac{75}{256}\left(\frac{r}{b}\right)^7 P_7(\cos\theta) + \dots\right]$$

Which also matches our expectation. □